A STRATIFICATION OF SOME MODULI SPACES OF COHERENT SYSTEMS ON ALGEBRAIC CURVES AND THEIR HODGE–POINCARÉ POLYNOMIALS

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ABSTRACT. When $k < n$, we study the coherent systems that come from a BGN extension in which the quotient bundle is strictly semistable. In this case we describe a stratification of the moduli space of coherent systems. We also describe the strata as complements of determinantal varieties and we prove that these are irreducible and smooth. These descriptions allow us to compute the Hodge polynomials of this moduli in some cases. In particular, we give explicit computations for the case in which $n-k=2$. From these computations one may obtain the usual Poincaré polynomial when $(n,d,k)=1$ and $(n-k,d)=(2,d)\neq 1$.

1. INTRODUCTION AND STATEMENT OF RESULTS

A coherent system of type (n,d,k) on an algebraic curve X of genus g which is smooth and projective, consists of a pair (E,V) where E is a vector bundle on X of rank n and degree d and V is a subspace of dimension k of sections of E. Coherent systems were introduced by J. Le Potier [LeP1], and N. Raghavendra and P. A. Vishwanath [RV]. Among other things, the study of coherent systems is interesting due to its relation to the Brill–Noether problem for higher rank and some results in this direction can be seen in [BG2]. Its relation to gauge theory, for instance the fact that the α-stability condition is equivalent to the existence of solutions to a certain set of gauge theoretic equations, one of which is essentially the vortex equation (see [BG1]). Coherent systems are also a generalisation of linear series on algebraic curves.

For these objects there is a notion of stability that depends on a real parameter α. A coherent subsystem (E',V') is a subbundle E' of E together with a subspace of sections $V'\subset H^0(X,E')\cap V$. One defines the α-slope as $\mu_\alpha(E,V)=\frac{d}{n}+\alpha\frac{k}{n}$. The coherent system is called α-semistable (resp. α-stable) if the α-slope of every coherent subsystem is less than or equal to (resp. smaller than) the α-slope of the coherent system.

Using the notion of α-(semi)stability, A. King and P. E. Newstead (see [KN]) constructed a GIT quotient for these objects. They proved that for fixed n, d, k and α, there exists a projective scheme $\widetilde{G}(\alpha;n,d,k)$ which is a coarse moduli space of α-semistable coherent systems of type (n,d,k). Let $G(\alpha;n,d,k)$ be the moduli space of α-stable coherent systems of the given type.

In recent years these moduli spaces have been broadly studied by S. B. Bradlow, O. García-Prada, V. Mercat, V. Muñoz and P. E. Newstead (see [BGMN], [BGMMN] and [BGMMN2]) for genus greater

1991 *Mathematics Subject Classification.* 14H60, 14D20, 14F45.

Key words and phrases. Coherent systems, moduli spaces, vector bundles, stratification, Hodge polynomials.

This work has been partially supported by two EC Training Fellowships. The first one was within the "Liverpool Mathematics International Training Site" (LIMITS) supported as a Marie Curie Training Site of the European Community Programme "Improving Human Research Potential and the Socio-Economic Knowledge Base" Contract No. HPMT-CT-2001-00277. The second one was within the Research Training Network LIEGRITS: Flags, Quivers and Invariant Theory in Lie Representation Theory, which is a Marie Curie Research Training Network funded by the European community as project MRTN-CT 2003-505078.

than or equal to two, and by H. Lange and P. E. Newstead for genus zero and one (see [LN1], [LN2], [LN3] and [LN4]).

In this paper we deal with the cases in which $g \geq 2$, $k < n$ and α "large". Under these hypotheses the moduli space $G_L(n,d,k)$ of α-stable coherent systems for "large" α is birationally equivalent to a Grassmannian fibration over $\mathcal{M}(n-k,d)$ (see Proposition 4.5), where $\mathcal{M}(n,d)$ denotes the moduli space of stable bundles of rank n and degree d on X. This is given by the observation that a coherent system (E,V) of fixed type (n,d,k) corresponds to a certain extension of the form (BGN extension, see Definition 4.1)

$$0 \to \mathcal{O}^{\oplus k} \to E \to F \to 0.$$

This is used in [BGMMN] to obtain some information on the geometry and the cohomology of these moduli spaces; in particular, some Betti numbers, fundamental groups and flip loci are computed.

However, there is not a good enough geometric description of these moduli spaces. The results in [BGMMN] do not cover fully the cases in which the coherent system comes from a BGN extension in which the quotient bundle F is strictly semistable. In this article we study these cases giving a stratification of these moduli spaces by looking at the quotient bundle F. We also study their Hodge polynomials.

The layout of the paper is as follows. In Section 2 we introduce some general theory of extensions of vector bundles; Subsection 2.1 is a review of the results on extensions that can be found in [Mu], while Subsection 2.2 is a review of the theory described in [L] of universal families of extensions. In Section 3 we give a summary of the results and definitions about coherent systems that can be found in [BGMN] and [BGMMN].

In Section 4 we study the BGN extensions and we give the conditions that a BGN extension must satisfy in order to contradict α-stability (Theorem 4.6). In Section 5 we estimate the codimension of the variety of semistable vector bundles such that the coherent system that they induce is not α-stable (Theorem 5.1). In Section 6 we study the sets that classify the quotient bundles that appear in the BGN extensions associated to our coherent systems. To do that, from the results in Section 4 we must look at the Jordan–Hölder filtrations that are admitted by a given F. Then, we study the possible sets of these filtrations and we give geometric descriptions of them in terms of sequences of projective fibrations (see Proposition 6.5 for a general contruction). We also estimate the number of parameters on which these sets depend. This description will allow us in Section 7 to construct a stratification of the moduli space of coherent systems for $n < k$ in some cases (Theorem 7.2). We also describe these strata as complements of determinantal varieties (Theorem 7.5) and we prove that they are smooth and irreducible (Theorem 7.10). We finish this paper studying the Hodge polynomials of these moduli spaces. We start Section 8 by giving a review of Hodge theory and the relationship between Hodge–Deligne and Hodge–Poincaré polynomials that we denote by $\mathcal{H}$ and HP respectively. For a complex algebraic variety X, not necessarily smooth, compact or irreducible, we define its *Hodge–Deligne polynomial* (or virtual Hodge polynomial) as

$$\mathcal{H}(X)(u,v) = \sum_{p,q} (-1)^{p+q} \chi^c_{p,q}(X) u^p v^q \in \mathbb{Z}[u,v],$$

and its *Hodge–Poincaré polynomial* as

$$HP(X)(u,v) = \sum_{p,q} (-1)^{p+q} \chi_{p,q}(X) u^p v^q = \sum_{p,q,k} (-1)^{p+q+k} h^{p,q}(H^k(X)) u^p v^q.$$

Here the Euler characteristics that we consider, $\chi^c_{p,q}$ and $\chi_{p,q}$ respectively, are the sums of the dimensions of certain filtrations associated to the cohomology groups with compact support and to the usual cohomology groups for the Hodge–Deligne polynomials and the Hodge–Poincaré polynomials respectively. We also introduce equivariant Hodge–Poincaré polynomials and we study how to compute the Hodge–Poincaré

polynomials of the strata in a general setup (see Theorem 8.17). We conclude the paper by giving explicit computations of the cases in which $n-k=2$. These are the following. The Hodge–Deligne polynomial of the moduli space $G_L(n,d,k)$ for $(n-k,d)=(2,d)=1$ is (see Theorem 8.19)

$$\begin{aligned}\mathcal{H}(G_L(n,d,k))(u,v) =&(1+u)^g(1+v)^g\cdot\frac{(1+u^2v)^g(1+uv^2)^g-u^gv^g(1+u)^g(1+v)^g}{(1-uv)(1-u^2v^2)}\cdot\\ &\cdot\frac{(1-(uv)^{2(g-1)+d-k+1})\cdot\ldots\cdot(1-(uv)^{2(g-1)+d})}{(1-uv)\cdot\ldots\cdot(1-(uv)^k)}\end{aligned}$$

and when $(n-k,d)=(2,d)\neq 1$ it is computed in Theorem 8.20. From the latter theorem one can obtain the usual Poincaré polynomial of $G_L(n,d,k)$ when $(n,d,k)=1$ and $(n-k,d)=(2,d)\neq 1$ just by writing $u=v$.

2. Some theory of vector bundles and their extensions

2.1. **General theory of extensions.** In this subsection we introduce some general theory about extensions of vector bundles as well as a classification theorem. All these results are known and can be found for example in [Mu].

Consider the following short exact sequence of vector bundles

$$\xi: 0\to M\to E\to L\to 0. \tag{1}$$

Tensoring with the dual bundle $L^\vee$ gives a short exact sequence

$$0\to\mathscr{H}om(L,M)\to\mathscr{H}om(L,E)\to\mathscr{E}nd(L)\to 0, \tag{2}$$

and its associated long exact sequence of vector spaces is

$$\begin{aligned}0\to Hom(L,M)\to Hom(L,E)\to End(L) &\xrightarrow{\delta}\\ \xrightarrow{\delta} H^1(\mathscr{H}om(L,M))\to H^1(\mathscr{H}om(L,E))&\to H^1(\mathscr{E}nd(L))\to 0.\end{aligned}$$

Definition 2.1. The image under the coboundary map δ of $id\in End(L)$, which we will denote by

$$\overline{\xi}=\delta(id)\in H^1(\mathscr{H}om(L,M)),$$

is called *the extension class* of the exact sequence (1).

By construction, if $\overline{\xi}=0$, then there exists a homomorphism $f: L\to E$ for which the composition $L\xrightarrow{f}E\to L$ (where the second map is the surjection of (1)) is the identity homomorphism of L. In other words, the sequence (1) splits. In particular, we have

Proposition 2.2. *If $H^1(\mathscr{H}om(L,M))=0$, then every exact sequence (1) splits.*

Now, given vector bundles L and M, we will give a theorem that classifies the bundles E having M as a subbundle with quotient L. First of all

Definition 2.3.

(i) A short exact sequence (1)

$$\xi: 0\to M\to E\to L\to 0.$$

is called an *extension* of L by M.

(ii) Two extensions ξ and ξ' are *equivalent* if there exists an isomorphism of vector bundles $f : E \xrightarrow{\sim} E'$ and a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

Note that the extension class $\overline{\xi}$ defined in Definition 2.1 depends only on the equivalence class of the extension in this sense. We have the theorem

Theorem 2.4. *The assignment*

$$\{\text{extensions of } L \text{ by } M\}/\text{equivalence} \to H^1(\mathscr{H}om(L,M))$$

given by $\xi \mapsto \overline{\xi}$ (Definition 2.1) is a bijection.

2.2. **Universal families of extensions.** Here we introduce some theory of universal families of extensions, the conditions for the existence of global universal families are given as well as the conditions for the existence of universal families in a "local" sense. All these results can be found in [L].

Let $f : X \to Y$ be a flat projective morphism of noetherian schemes and $\mathscr{F}$ and $\mathscr{G}$ coherent $\mathcal{O}_X$-modules, flat over Y. Let $Ext^1_X(\mathscr{F},\mathscr{G})$ be the vector space parametrizing the extensions of $\mathscr{F}$ by $\mathscr{G}$ over X.

In order to construct such universal families of extensions we need to introduce the ith relative Ext-sheaf. This relative sheaf is defined as $\mathscr{E}xt^i_f(\mathscr{F},\mathscr{G}) := R^i(f_*\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\bullet))(\mathscr{G})$. It is easy to prove that:

(i) $\mathscr{E}xt^i_f(\mathscr{F},\mathscr{G})$ is the sheaf associated to the presheaf

$$U \to Ext^i_{f^{-1}(U)}(\mathscr{F}_{|f^{-1}(U)},\mathscr{G}_{|f^{-1}(U)})$$

on Y.

(ii) If $\mathscr{F}$ is locally free

$$\mathscr{E}xt^i_f(\mathscr{F},\mathscr{G}) \simeq R^i f_*(\mathscr{F}^\vee \otimes \mathscr{G}).$$

We restrict ourselves to the case in which $\mathscr{F}$ is locally free. In this case, for every coherent sheaf $\mathscr{C}$ on X and for every point $y \in Y$, the usual base change homomorphism

$$\tau^i(y) : R^i f_*\mathscr{C} \otimes k(y) \to H^i(X_y,\mathscr{C}_y)$$

is the homomorphism

$$\varphi^i(y) : \mathscr{E}xt^i_f(\mathscr{F},\mathscr{G}) \otimes k(y) \to Ext^i_{X_y}(\mathscr{F}_y,\mathscr{G}_y). \tag{3}$$

Moreover the "Cohomology and Base Change" theorem ([H, III. §12. Theorem 12.11]) applied to $\varphi^i(y)$ tells us that if $\varphi^i(y)$ is surjective then (i) there is a neighbourhood U of y such that $\varphi^i(y')$ is an isomorphism for all $y' \in U$; and (ii) $\varphi^{i-1}(y)$ is surjective if and only if $\mathscr{E}xt^i_f(\mathscr{F},\mathscr{G})$ is locally free in a neighbourhood of y.

Definition 2.5. If $\varphi^i(y)$ is an isomorphism for all $y \in Y$ we say that $\mathscr{E}xt^i_f(\mathscr{F},\mathscr{G})$ *"commutes with base change"*.

We will define now what a family of extensions is. For every point $y \in Y$ let

$$\phi_y : Ext^1_X(\mathscr{F},\mathscr{G}) \to Ext^1_{X_y}(\mathscr{F}_y,\mathscr{G}_y),$$

be the map that assigns to every extension class of $\mathscr{F}$ by $\mathscr{G}$, the extension class of $\mathscr{F}_y$ by $\mathscr{G}_y$.

Definition 2.6. A *family of extensions* of $\mathscr{F}$ by $\mathscr{G}$ over Y is a family $(e_y \in Ext^1_{X_y}(\mathscr{F}_y, \mathscr{G}_y))_{y\in Y}$ such that there is an open covering $\mathscr{U} = (U_i)_{i\in I}$ of Y and for each $i \in I$ an element $\sigma_i \in Ext^1_{f^{-1}(U_i)}(\mathscr{F}_{|f^{-1}(U_i)}, \mathscr{G}_{|f^{-1}(U_i)})$ such that $e_y = \phi_{i,y}(\sigma_i)$ for every $y \in Y$. Here $\phi_{i,y}$ denotes the canonical map

$$Ext^1_{f^{-1}(U_i)}(\mathscr{F}_{|f^{-1}(U_i)}, \mathscr{G}_{|f^{-1}(U_i)}) \to Ext^1_{X_y}(\mathscr{F}_y, \mathscr{G}_y).$$

The family of extensions is called *globally defined* if the covering $\mathscr{U}$ may be taken to be Y itself.

In order to study the universal families of extensions, it will be fundamental to have a relationship between the relative Ext's, these are the groups $\mathscr{E}xt^i_f(\mathscr{F}, \mathscr{G})$, and the global ones, $Ext^j_X(\mathscr{F}, \mathscr{G})$. This relationship is accounted for by a spectral sequence (see for example [G]) whose E_2-term is given by $E_2^{p,q} = H^p(Y, \mathscr{E}xt^q_f(\mathscr{F}, \mathscr{G}))$ and which abuts to $Ext^*_X(\mathscr{F}, \mathscr{G})$. In particular we have an exact sequence

$$0 \longrightarrow H^1(Y, f_*\mathscr{H}om_X(\mathscr{F}, \mathscr{G})) \xrightarrow{\ \varepsilon\ } Ext^1_X(\mathscr{F}, \mathscr{G}) \xrightarrow{\ \mu\ }$$

$$\xrightarrow{\ \mu\ } H^0(Y, \mathscr{E}xt^1_f(\mathscr{F}, \mathscr{G})) \xrightarrow{\ d_2\ } H^2(Y, f_*\mathscr{H}om_X(\mathscr{F}, \mathscr{G})),$$

where ε and μ denote the edge homomorphisms. For a point $y \in Y$ let $i_y : \mathscr{E}xt^1_f(\mathscr{F}, \mathscr{G}) \to \mathscr{E}xt^1_f(\mathscr{F}, \mathscr{G}) \otimes k(y)$ denote the canonical homomorphism, what we have is that

$$\phi_y(e) = \varphi^1(y) i_y \mu(e)$$

for all $e \in Ext^1_X(\mathscr{F}, \mathscr{G})$ and for all $y \in Y$.

If we suppose in addition to the above hypotheses that Y is reduced and $\mathscr{E}xt^1_f(\mathscr{F}, \mathscr{G})$ commutes with base change, then there is a canonical bijection between the set of all globally defined families of extensions of $\mathscr{F}$ by $\mathscr{G}$ over Y and

$$Ext^1_X(\mathscr{F}, \mathscr{G})/H^1(Y, f_*\mathscr{H}om_X(\mathscr{F}, \mathscr{G})) \subseteq H^0(Y, \mathscr{E}xt^1_f(\mathscr{F}, \mathscr{G})).$$

There is also a canonical bijection between the set of all families of extensions of $\mathscr{F}$ by $\mathscr{G}$ over Y and $H^0(Y, \mathscr{E}xt^1_f(\mathscr{F}, \mathscr{G}))$. Moreover, if $H^2(Y, f_*\mathscr{H}om_X(\mathscr{F}, \mathscr{G}))$ vanishes, $H^0(Y, \mathscr{E}xt^1_f(\mathscr{F}, \mathscr{G}))$ parametrizes the globally defined families of extensions.

Suppose in addition to the general hypotheses that $\mathscr{E}xt^i_f(\mathscr{F}, \mathscr{G})$ commutes with base change for $i = 0$, 1. Let $q_S : X \times_Y S \to X$ and $p_S : X \times_Y S \to S$ be the projections. Then the functor

$$E(S) := H^0(S, \mathscr{E}xt^1_{p_S}(q_S^*\mathscr{F}, q_S^*\mathscr{G}))$$

of the category of noetherian Y-schemes to the category of sets, is a contravariant functor that is representable by the vector bundle $V = \mathbb{V}(\mathscr{E}xt^1_f(\mathscr{F}, \mathscr{G})^\vee)$ over Y associated to the locally free sheaf $\mathscr{E}xt^1_f(\mathscr{F}, \mathscr{G})^\vee$.

Proposition 2.7. *Suppose Y is reduced and $\mathscr{E}xt^i_f(\mathscr{F}, \mathscr{G})$ commutes with base change for $i = 0$, 1. Then there is a family $(e_v)_{v\in V}$ of extensions of $q_V^*\mathscr{F}$ by $q_V^*\mathscr{G}$ over V which is universal in the category of reduced noetherian Y-schemes.*

2.8. Here "universal" means: Given a reduced noetherian Y-scheme S and a family of extensions $(e_s)_{s\in S}$ of $q_S^*\mathscr{F}$ by $q_S^*\mathscr{G}$ over S, then there is exactly one morphism $g : S \to V$ over Y such that $(e_s)_{s\in S}$ is the pull-back of $(e_v)_{v\in V}$ by g.

There exists a projective analogue of the above result. Under the same hypotheses as in the last proposition, consider the functor

$$PE(S) := \text{set of invertible quotients of } \mathscr{E}xt^1_{p_S}(q_S^*\mathscr{F}, q_S^*\mathscr{G})^\vee$$

of the category of noetherian Y-schemes to the category of sets, where q_S and p_S are as above. This is a contravariant functor that is representable by the projective bundle $P = \mathbb{P}(\mathscr{E}xt^1_f(\mathscr{F},\mathscr{G})^\vee)$ over Y associated to the locally free sheaf $\mathscr{E}xt^1_f(\mathscr{F},\mathscr{G})^\vee$.

Proposition 2.9. *Suppose Y is reduced and $\mathscr{E}xt^i_f(\mathscr{F},\mathscr{G})$ commutes with base change for $i = 0$, 1. Then there is a family $(e_p)_{p\in P}$ of extensions of $q_P^*\mathscr{F}$ by $q_P^*\mathscr{G} \otimes p_P^*\mathcal{O}_P(1)$ over P which is universal in the category of reduced noetherian Y-schemes for the classes of families of nonsplitting extensions of $q_P^*\mathscr{F}$ by $q_P^*\mathscr{G} \otimes p_P^*\mathscr{L}$ over S with arbitrary $\mathscr{L} \in Pic(S)$ modulo the canonical operation of $H^0(S, \mathcal{O}_S^\vee)$.*

In particular, as a restriction of these results we obtain the classical ones on universal extensions, here "universal" is in the usual sense. These are the following (see [NR], [R] and the Appendix on extensions of [S]):

Fix an algebraic variety X, and let S and T be two more algebraic varieties. Let V (resp. W) be a vector bundle on $S \times X$ (resp. $T \times X$), such that $\dim(H^1(X, \mathscr{H}om(W_t, V_s)))$ is independent of the point (s,t) of $S \times T$. Let $p_{S\times T}$, p_T and p_S be the projections $S \times T \times X \to S \times T$, $S \times T \to T$ and $S \times T \to S$ respectively.

Let
$$F = R^1(p_{S\times T})_*(\mathscr{H}om((p_T \times id_X)^*W, (p_S \times id_X)^*V)).$$
This is a vector bundle on $S \times T$. Let $\pi : F \to S \times T$ be the projection.

Proposition 2.10. *If*
$$h^i(S \times T, (p_{S\times T})_*(\mathscr{H}om((p_T \times id_X)^*W, (p_S \times id_X)^*V) \otimes F^\vee) = 0$$
for $i = 1$, 2, there exists a vector bundle E on $F \times X$ and an exact sequence
$$0 \to (\pi \times id_X)^*(p_S \times id_X)^*V \to E \to (\pi \times id_X)^*(p_T \times id_X)^*W \to 0,$$
such that for every point $(s,t) \in S\times T$ and every element $h \in F_{(s,t)} = H^1(X, \mathscr{H}om(W_t, V_s))$, its restriction to $\{h\} \times X$:
$$0 \to V_s \to E_h \to W_t \to 0$$
is the extension associated to h.

As in the general case, we have a projective analogue of this proposition.

Remark 2.11. The hypotheses of Proposition 2.10 are verified in the following cases:

(a) When for all $(s,t) \in S \times T$, we have that $Hom(W_t, V_s) = \{0\}$.
(b) When S and T are affine.

3. Coherent systems

In this section we introduce some general theory on coherent systems on algebraic curves. This material is a summary of results that can be found in [BG2], [BGMN] and [BGMMN].

Let X be a smooth projective algebraic curve of genus greater than or equal to 2.

Definition 3.1. A *coherent system* on X of type (n,d,k) is a pair (E,V), where E is a vector bundle on X of rank n and degree d and V is a subspace of dimension k of the space of sections $H^0(E)$.

Definition 3.2. Fix $\alpha \in \mathbb{R}$. Let (E,V) be a coherent system of type (n,d,k). The *α-slope* of (E,V), $\mu_\alpha(E,V)$, is defined by
$$\mu_\alpha(E,V) = \frac{d}{n} + \alpha \cdot \frac{k}{n}.$$

We say that (E, V) is *α-stable* if

$$\mu_\alpha(E', V') < \mu_\alpha(E, V)$$

for all proper subsystems (E', V') (i.e. for every non-zero subbundle E' of E and every subspace $V' \subseteq V \cap H^0(E')$ with $(E', V') \neq (E, V)$). Analogously *α-semistability* is defined by changing $<$ to $\leqslant$.

There exists a (coarse) moduli space for α-stable coherent systems of type (n, d, k) which we denote by $G(\alpha; n, d, k)$.

Definition 3.3. We say that $\alpha > 0$ is a *critical value* if there exists a proper subsystem (E', V') such that $\frac{k'}{n'} \neq \frac{k}{n}$ but $\mu_\alpha(E', V') = \mu_\alpha(E, V)$. We also regard 0 as a critical value.

For α not critical, if $\gcd(n, d, k) = 1$, the α-semistability condition and the α-stability condition are equivalent. For $k < n$, it is easy to see that there are finitely many critical values, this is also true when $k \geq n$.

If we label the critical values of α by α_i, starting with $\alpha_0 = 0$, we get a partition of the α-range into a set of intervals (α_i, α_{i+1}). Within the interval (α_i, α_{i+1}) the property of α-stability is independent of α, that is if α, $\alpha' \in (\alpha_i, \alpha_{i+1})$ then $G(\alpha; n, d, k) = G(\alpha'; n, d, k)$. We shall denote this moduli space by G_i.

Suppose now that $G(\alpha; n, d, k) \neq \emptyset$ for at least one value of α.

Proposition 3.4. *Let $k < n$ and let α_L be the biggest critical value smaller than $\frac{d}{n-k}$. The α-range is divided into a finite set of intervals determined by critical values*

$$0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_L < \frac{d}{n-k}.$$

If $\alpha > \frac{d}{n-k}$, the moduli spaces are empty.

The difference between adjacent moduli spaces in the family G_0, G_1, ..., G_L is accounted for by the subschemes $G_i^+ \subseteq G_i$ and $G_i^- \subseteq G_{i-1}$, where G_i^+ consists of all (E, V) in G_i which are not α-stable if $\alpha < \alpha_i$ and $G_i^- \subseteq G_{i-1}$ contains all (E, V) in G_{i-1} which are not α-stable if $\alpha > \alpha_i$. It follows that $G_i - G_i^+ = G_{i-1} - G_i^-$ and that G_i is transformed into G_{i-1} by removal of G_i^+ and the insertion of G_i^-.

Definition 3.5. We refer to such a procedure as a *flip*. We call the subschemes $G_i^\pm$ *the flip loci*. We say that a flip is *good* if the flip loci have strictly positive codimension in every component of the moduli spaces G_i and G_{i-1} respectively. Under these conditions the moduli spaces are birationally equivalent.

4. Study of the BGN extensions

When $k < n$ we denote by $G_L(n, d, k)$ the moduli space of coherent systems of type (n, d, k) for α large, i.e., $\alpha_L < \alpha < \frac{d}{n-k}$.

Definition 4.1 ([BG2, BGN])**.** A *BGN extension* is an extension of vector bundles

$$0 \to \mathcal{O}^{\oplus k} \to E \to F \to 0$$

which satisfies the following conditions:

(i) $\operatorname{rank} E = n > k$,
(ii) $\deg E = d > 0$,
(iii) $H^0(F^*) = 0$,
(iv) if $e = (e_1, \ldots, e_k) \in H^1(F^* \otimes \mathcal{O}^{\oplus k}) = H^1(F^*)^{\oplus k}$ denotes the class of the extension, then $e_1, \ldots, e_k$ are linearly independent as vectors in $H^1(F^*)$.

Definition 4.2. Two BGN extensions are *equivalent* if one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^{\oplus k'} & \longrightarrow & E' & \longrightarrow & F' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}^{\oplus k} & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

where the vertical arrows are isomorphisms, in particular $k = k'$. An equivalence class of BGN extensions will be called a *BGN extension class.*

Proposition 4.3 ([BG2],[BGMN]). *Suppose that $0 < k < n$ and $d > 0$. Let $\alpha_L < \alpha < \frac{d}{n-k}$. Let (E, V) be an α-semistable coherent system of type (n, d, k). Then (E, V) defines a BGN extension class represented by an extension*

$$0 \to \mathcal{O}^{\oplus k} \to E \to F \to 0$$

with F semistable. Conversely, any BGN extension of type (n, d, k) in which the quotient F is stable gives rise to an α-stable coherent system of the same type.

Remark 4.4. Note that if in the last part of Proposition 4.3 our quotient bundle F is only semistable, the coherent system can fail to be α-stable or even α-semistable.

Proposition 4.5 ([BG2],[BGMN],[BGMMN]). *Suppose $n \geq 2$ and $0 < k \leq n$. Then $G_L(n, d, k) \neq \emptyset$ if and only if*

$$d > 0, \;\; k \leq n + \frac{1}{g}(d - n) \;\; \text{and} \;\; (n, d, k) \neq (n, n, n),$$

and it is then always irreducible and smooth of dimension $\beta(n, d, k) = n^2(g-1) + 1 - k(k - d + n(g-1))$.

If $0 < k < n$, $G_L(n, d, k)$ is birationally equivalent to a fibration over the moduli space of semistable vector bundles, $\mathcal{M}(n-k, d)$ with fibre the Grassmannian $\mathrm{Gr}(k, d + (n-k)(g-1))$. More precisely, if W denotes the subvariety of $G_L(n, d, k)$ consisting of coherent systems for which the quotient bundle F is strictly semistable, then $G_L(n, d, k) \setminus W$ is isomorphic to a Grassmann fibration over $\mathcal{M}(n-k, d)$.

If in addition $\gcd(n-k, d) = 1$, then $W = \emptyset$ and $G_L(n, d, k) \to \mathcal{M}(n-k, d)$ is the Grassmann fibration associated to some vector bundle over $\mathcal{M}(n-k, d)$.

Our next goal is to study what happens when the quotient bundle F is strictly semistable. To this end, consider a BGN extension as above:

$$0 \to \mathcal{O}^{\oplus k} \to E \to F \to 0, \tag{4}$$

in which F is strictly semistable with rank $(n-k)$ and degree $d > 0$. Let

$$(e_1, \ldots, e_k) \in H^1(F^* \otimes \mathcal{O}^{\oplus k}) = H^1(F^*)^{\oplus k}$$

be the class of the extension of E, with $\{e_i\}_i$ linearly independent as vectors in $H^1(F^*)$.

Let (E, V) be the coherent system corresponding to the extension (4). Consider a subsystem (E', V'). In general, this subsystem determines an extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W' & \longrightarrow & E' & \longrightarrow & F' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \hookdownarrow & & \\ 0 & \longrightarrow & \mathcal{O}^{\oplus k} & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

with F' a subsheaf of F, W' a subbundle of $\mathcal{O}^{\oplus k}$. Let $\alpha \in (\alpha_L, \frac{d}{n-k})$ be sufficiently close to $\frac{d}{n-k}$. We are going to study the relationship between $\mu_\alpha(E', V')$ and $\mu_\alpha(E, V)$. It is proved in [BG2] (Lemma 4.3) that for an extension

$$0 \to W' \to E' \to F' \to 0,$$

$\deg(W') \leq 0$ and $= 0$ if and only if $W' \cong \mathcal{O}^{\oplus k'}$. Moreover $h^0(W') \leq \mathrm{rank}(W')$ and $=$ if and only if $W' \cong \mathcal{O}^{\oplus k'}$. This can be proved by considering the vector bundle generated by the global sections of W' and bearing in mind that if $h^0(W') = \mathrm{rank}(W')$ and $W' \ncong \mathcal{O}^{\oplus k'}$ then the degree of W' would be positive, which contradicts the fact that $\deg(W') \leq 0$.

We divide the study into the following cases:

§. F' proper, non-trivial subsheaf and $W' \ncong \mathcal{O}^{\oplus k'}$. Let $l' = \mathrm{rank}(W')$ and $k' = h^0(W')$. Since $\deg(W') \leq 0$ we have

$$\mu(E') = \frac{n'-l'}{n'} \cdot \frac{(\deg W' + \deg F')}{n'-l'} \leq \frac{\deg F'}{n'-l'} \cdot \frac{n'-l'}{n'} \leq \mu(F) \cdot \frac{n'-l'}{n'}.$$

Following the computations of [BG2] page 139, we have

$$\mu_\alpha(E',V') - \mu_\alpha(E,V) \leq \frac{\varepsilon}{n-k}\left[\frac{k}{n} - \frac{k'}{n'}\right] + \mu(F)\frac{k'-l'}{n'},$$

where $\epsilon = d - \alpha(n-k) > 0$ and we know that $d > 0$ and $\mu(F) > 0$. Lemma 4.3 of [BG2] implies that $k' < l'$, so choosing ε sufficiently small -note that in Section 3 we saw that the condition of α-stability does not change within an interval (α_i, α_{i+1}), i.e., $G(\alpha; n, d, k) = G(\alpha'; n, d, k)$ for all $\alpha, \alpha' \in (\alpha_i, \alpha_{i+1})$- we have

$$\frac{\varepsilon}{n-k}\left[\frac{k}{n} - \frac{k'}{n'}\right] + \mu(F)\frac{k'-l'}{n'} < 0$$

and we conclude that (E',V') does not contradict the α-stability of (E,V).

§. The cases: $F' = 0$; $F' = F$ and $W' \ncong \mathcal{O}^{\oplus k'}$; $F' = F$ and $W' = \mathcal{O}^{\oplus k'}$. These cases follow in the same way as Theorem 4.2 of [BG2], and (E',V') does not contradict the α-stability of (E,V).

§. F' a proper, non-trivial subsheaf and $W' = \mathcal{O}^{\oplus k'}$. We know that $\deg(W') = 0$, and (E',V') is a subsystem of type (n',d',k') where:

$$d' = \deg F',\ n' = k' + \mathrm{rank} F' \text{ and } \mu(F') \leq \mu(F) \Rightarrow \frac{d'}{n'-k'} \leq \frac{d}{n-k}.$$

So, bearing in mind that $\alpha \in (\alpha_L, \frac{d}{n-k})$ sufficiently close to $\frac{d}{n-k}$, let $\alpha = \frac{d}{n-k} - \frac{\varepsilon}{n-k}$ with ε sufficiently small, we have

$$\begin{aligned}\mu_\alpha(E',V') - \mu_\alpha(E,V) &= \frac{d'}{n'} - \frac{d}{n} + \alpha(\frac{k'}{n'} - \frac{k}{n}) = \\ &= \frac{1}{n'(n-k)}[nd' - dn' - (kd' - dk')] + \frac{\varepsilon}{n-k}\left[\frac{k}{n} - \frac{k'}{n'}\right]. \qquad (5)\end{aligned}$$

In the case in which $\frac{d}{n-k} = \mu(F) > \mu(F') = \frac{d'}{n'-k'}$, we have $nd' - dn' - (kd' - dk') < 0$, so choosing ε properly, this does not contradict the α-stability of (E,V). In the other case, when $\mu(F) = \mu(F')$, we obtain

$$\mu_\alpha(E',V') - \mu_\alpha(E,V) = \frac{\varepsilon}{n-k}\left[\frac{k}{n} - \frac{k'}{n'}\right].$$

So the α-(semi)stability depends on whether $\frac{k}{n} - \frac{k'}{n'}$ is $>$, $=$ or $<$ than 0. Hence, we only have trouble when $\frac{k}{n} \geq \frac{k'}{n'}$.

If we study the relationship between the invariants k, k', n and n', we find that

$$\mathrm{rank}(F') = n' - k' < \mathrm{rank}(F) = n - k,$$

because F' is a proper subbundle of F.

From now on, we will study the cases in which the coherent system that comes from our original BGN extension fails to be α-stable. These cases are those in which we have a coherent subsystem (E',V') of type (n',d',k') such that $n-k>n'-k'>0$, $\frac{k}{n}\geq\frac{k'}{n'}$ and $\mu(F)=\mu(F')$. Note that the condition $\frac{k}{n}\geq\frac{k'}{n'}$ is equivalent to $\frac{k}{n-k}\geq\frac{k'}{n'-k'}$. We can restrict ourselves to the case in which our extension

$$0\to\mathcal{O}^{\oplus k'}\to E'\to F'\to 0 \tag{6}$$

is either an extension verifying the properties (i)-(iii) of the definition of BGN extension (Definition 4.1) or, for the smallest value of k' for which the extension (6) exists, a BGN extension. This is proved in

Theorem 4.6. *A BGN extension (4) fails to be α-stable for $\alpha_L<\alpha<\frac{d}{n-k}$ if and only if there exists a BGN extension (6) and a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}^{\oplus k'} \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d] & F' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}^{\oplus k} \arrow[r] & E \arrow[r] & F \arrow[r] & 0
\end{tikzcd} \tag{7}$$

such that

$$n-k>\mathrm{rank}F'>0,\ \frac{k}{n-k}\geqslant\frac{k'}{\mathrm{rank}F'}\ \text{and}\ \mu(F)=\mu(F'). \tag{8}$$

Proof. The existence of (7) immediately implies that (4) is not α-stable. Conversely, if (4) is not α-stable, then there exists a diagram (7) for which (8) holds. We need only show that we can choose (7) so that (6) is a BGN extension.

Now (7) and (8) immediately imply conditions (i) and (ii) of Definition 4.1 for the extension (6). Moreover, since F is semistable, so is F'; since $\deg F'>0$ this implies that $H^0(F'^*)=0$, giving (iii).

Condition (iv), however, is not automatic. Let $(e_1,\dots,e_k)$ be the k-tuple classifying (4) and let $(e'_1,\dots,e'_k)$ be the image of this k-tuple under the surjection $H^1(F^*)\twoheadrightarrow H^1(F'^*)$. Put $k''=\dim<e'_1,\dots,e'_k>$. The existence of (7) implies that $k'\geqslant k''$. On the other hand, after applying an automorphism of $\mathcal{O}^{\oplus k}$, we can assume that $e'_{k''+1}=\ldots=e'_k=0$ and hence deduce the existence of a subextension of (4) of the form

$$0\to\mathcal{O}^{\oplus k''}\to E''\to F'\to 0 \tag{9}$$

classified by $(e'_1,\dots,e'_{k''})$. This is a BGN extension.

Moreover

$$\mathrm{rank}E''-k''=\mathrm{rank}F'=\mathrm{rank}E'-k'<n-k$$

and

$$k(\mathrm{rank}E''-k'')=k(\mathrm{rank}E'-k')\geq k'(n-k)\geq k''(n-k),$$

so $\frac{k}{n-k}\geqslant\frac{k''}{\mathrm{rank}F'}$ and the extension (9) satisfies all the conditions required for (6). ☐

5. The codimension of the "bad" part

We estimate the codimension of the subvariety of $\oplus^k H^1(F^*)$ whose elements are "bad", in the sense that the coherent systems they induce are not α-stable. We call this subvariety S.

Theorem 5.1. *Suppose that F has only finitely many subbundles F' with $\mu(F)=\mu(F')$. Then, the co-dimension of the subvariety S of $H^1(F^*)^{\oplus k}$ satisfies*

$$\mathrm{codim}(S)\geq\min\{((g-1)n'-k'g+d')(k-k')\}, \tag{10}$$

where this minimum is taken over all the invariants n', d', k' for which F possesses a subbundle F' of type $(n'-k', d')$ satisfying

$$n-k > n'-k' > 0,\ \frac{k}{n} \geq \frac{k'}{n'} \text{ and } \frac{d}{n-k} = \frac{d'}{n'-k'}. \tag{11}$$

Proof. By Theorem 4.6 the BGN extensions that could give us any trouble are those which possess BGN subextensions of the form (6). Bearing this in mind, to calculate the codimension we consider the following picture

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}^{\oplus k'} \arrow[r] \arrow[d, hook] & E' \arrow[r] \arrow[d] & F' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}^{\oplus k} \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & F' \arrow[r] \arrow[d, hook, "g"] & 0 \\
0 \arrow[r] & \mathcal{O}^{\oplus k} \arrow[r] & E \arrow[r] & F \arrow[r] & 0
\end{tikzcd} \tag{12}$$

where the first horizontal extension is a BGN subextension of our original BGN extension, which is the bottom one. We call now $(g^*e_1, \ldots, g^*e_k)$ the k-tuple image of the k-tuple $(e_1, \ldots, e_k)$ under the map $H^1(F^*) \twoheadrightarrow H^1(F'^*)$ induced by the canonical immersion g. The existence of the first horizontal extension tells us precisely that at most k' elements of $(g^*e_1, \ldots, g^*e_k)$ are linearly independent. Using Riemann-Roch and bearing in mind that $H^0(F'^*) = 0$, we get

$$h^1(F'^*) = (g-1)(n'-k') + d';$$

this identity tells us that the codimension of the subvariety $S_{F',k'}$ of $\oplus^k H^1(F^*)$, where the subindex F' refers to the subbundle F' of F, satisfies

$$\begin{aligned} \operatorname{codim}(S_{F',k'}) &\geq \big(h^1(F'^*) - k'\big)\big(k-k'\big) = \\ &= \big((g-1)n' - k'g + d'\big)\big(k-k'\big). \end{aligned} \tag{13}$$

So, if we look at all the subbundles F' of F for which $\mu(F') = \mu(F)$, we see the codimension of the subvariety of $H^1(F^*)^{\oplus k}$ of "bad" elements satisfies (10), where the minimum is taken over all the invariants n', d', k' satisfying (11). $\square$

Remark 5.2. Condition (iv) of Definition 4.1 tells us that

$$k \leq h^1(F^*) = d + (n-k)(g-1).$$

From this and (11)

$$\begin{aligned} k'(n-k) \leq k(n'-k') &\leq d(n'-k') + (n-k)(n'-k')(g-1) = \\ &= d'(n-k) + (n-k)(n'-k')(g-1) = \\ &= (n-k)h^1(F'^*). \end{aligned}$$

So $k' \leq h^1(F'^*)$, proving that there exist BGN extensions (6) and also that the lower bound for $\operatorname{codim} S$ is greater than or equal to 0.

Moreover, $k' = h^1(F'^*)$ is only possible if the above inequalities are equalities. In particular $\frac{k'}{n'} = \frac{k}{n}$, hence also $\frac{d'}{n'} = \frac{d}{n}$, and $k = h^1(F^*)$. Writing $\lambda = k/n$ and $\mu = d/n$, this means that

$$\lambda = 1 + \frac{1}{g}(\mu - 1) \tag{14}$$

(see [BGN] and [Me]). So (4) and (6) correspond to the same point in the Brill–Noether map of [BGN] and this point lies on the line given by (14).

Conversely, if the point corresponding to (4) lies on the line (14) and $\gcd(n,d,k) > 1$, we can find (n',d',k') with $\frac{k'}{n'} = \frac{k}{n}$, $\frac{d'}{n'} = \frac{d}{n}$ and $n'-k' < n-k$.

If this happens and F possesses a subbundle F' with invariants $(n'-k',d')$, then $h^1(F'^*) = k'$ and the diagram (7) exists, proving that the corresponding (E,V) is not α-stable. In this case there are no α-stable (E,V) with quotient F.

In all other cases, the general (E,V) is α-stable.

6. Some geometry of the spaces that classify the quotients

In Theorem 4.6 we saw that in order to find out if a coherent system is not α-stable we have to look at the quotient bundle that appears in its associated BGN extension. Those coherent systems that fail to be α-stable satisfy that their quotient bundle has subbundles with the same slope as the quotient bundle and that satisfy the properties described in the Theorem.

For a given vector bundle F, all the subbundles of F whose slope is the same as the slope of F, appear in some of the Jordan–Hölder filtrations of F. Bearing this in mind, in this section we study the sets of all possible Jordan–Hölder filtrations of a given vector bundle. From these sets we will define a stratification of $G_L(n,d,k)$.

We give some sort of "universal" constructions for these sets of Jordan–Hölder filtrations, some of them are described as projective fibrations, others are described in terms of "local" and global extensions, following the results and terminology of Subsection 2.2. All these geometrical descriptions will allow us in the following sections to describe our strata as complements of determinantal varieties and prove irreducibility and smoothness conditions for the strata.

First of all we need to introduce some definitions.

Definition 6.1. A *Jordan–Hölder filtration* of length r of a semistable vector bundle F is a filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset ... \subset F_r = F, \tag{15}$$

such that the quotients $Q_i = F_i/F_{i-1}$ are stable vector bundles satisfying $\mu(Q_i) = \mu(F)$ for $1 \le i \le r$.

It can be proved that every semistable vector bundle admits a Jordan–Hölder filtration and that all the Jordan–Hölder filtrations admitted by a given vector bundles have the same length. However, there is not a canonical Jordan–Hölder filtration associated to a semistable vector bundle F. Given a Jordan–Hölder filtration of a vector bundle, we may associate to it a canonical object. This is described in the following definition.

Definition 6.2. Consider the direct sum of the stable quotients $\mathrm{grad}(F) = \oplus_i Q_i$. We call $\mathrm{grad}(F)$ the *graded object* associated to F. This object is canonical in the sense that $\mathrm{grad}(F)$ is determined up to isomorphism by F (and hence Q_1, ...,Q_r are determined up to order).

In order to construct the stratification we look at the properties of the graded object associated to a given vector bundle. The main object we use is the type, its definition is the following.

Definition 6.3. We call the r-tuple $\underline{n} = (n_1,\dots,n_r) = (\mathrm{rank}(Q_1),\dots,\mathrm{rank}(Q_r))$ the *type* of the filtration (15). We denote by $\underline{n}(\sigma)$ the type $(n_{\sigma(1)},\dots,n_{\sigma(r)})$, where $\sigma \in S_r$, S_r being the group of permutation of r-elements.

We will use the type later on in this paper to define a stratification of the moduli space $G_L(n,d,k)$. Note that the type is not necessarily determined by F.

Consider now the Jordan–Hölder filtrations

$$0 = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_r = F, \tag{16}$$

where F is our usual strictly semistable vector bundle of rank $n-k$ and degree d. We begin by giving definitions and obtaining results for such a Jordan–Hölder filtration independently of the length r. We essentially provide conditions for such a filtration to be unique. Unfortunately, we don't have a description of the sets independently of the r. Later on in this section we restrict ourselves to the case r equals 2 and 3. In these cases we obtain complete answers and descriptions which allow us in the following section to obtain a stratification for $G_L(n,d,k)$ for the cases in which $n>k$ and $n-k$ equals 2 and 3.

We have the extensions

$$0 \to F_i \to F_{i+1} \to F_{i+1}/F_i \to 0, \tag{17}$$

and

$$0 \to F_i/F_{i-1} \to F_{i+1}/F_{i-1} \to F_{i+1}/F_i \to 0 \tag{18}$$

canonically associated to our Jordan–Hölder filtration (16). Here we denote $Q_i = F_i/F_{i-1}$ and let $\mathrm{rank}(Q_i) = n_i$ for all i.

Definition 6.4. We define $\mathcal{G}_{\underline{n}}$ as the set of Jordan–Hölder filtrations of type $\underline{n} = (n_1, \dots, n_r)$, such that the extensions (17) associated to the filtration are non-split for every i, and $Q_i \ncong Q_j$ for every $i \neq j$.

Proposition 6.5. *There is a sequence of projective fibrations for $\mathcal{G}_{\underline{n}}$, let*

$$\mathcal{G}_{\underline{n}} \to \mathcal{G}_{(n_1,\dots,n_{r-1})} \to \dots \to \mathcal{G}_{(n_1,n_2)} \to \mathcal{M}_1 \times \dots \times \mathcal{M}_r \backslash \Delta_r$$

where $\mathcal{M}_i$ is the moduli space of stable vector bundles of rank n_i and degree d_i and Δ_r is the "big diagonal", that is

$$\Delta_r := \{(Q_1, \dots, Q_r) \in \mathcal{M}_1 \times \dots \times \mathcal{M}_r \ \text{ such that } Q_i \cong Q_j \ \text{ for some } \ i \neq j\}.$$

In particular, when $\gcd(n_i, d_i) = 1$ for all i, $\mathcal{G}_{\underline{n}}$ parametrizes a universal filtration

$$0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_r.$$

Proof. We use induction. Let $\mathcal{M}_i = \mathcal{M}(n_i, d_i)$ be the moduli space of stable vector bundles of rank n_i and degree d_i. The construction depends on the existence of Poincaré bundles. In [R] it is proved that when $\gcd(n_i,d_i) \neq 1$ Poincaré bundles do not exist over $\mathcal{M}_i$. Then, we need to work at the Quot squeme level. Let $\mathcal{Q}_i$ be a certain Quot scheme. Let R_i^{ss} be the open set of $\mathcal{Q}_i$ of semistable points. If f_i is the morphism from R_i^{ss} to $\widetilde{\mathcal{M}}_i$, we have that $(\widetilde{\mathcal{M}}_i, f_i)$ is a good quotient of R_i^{ss}. Let $R_i^s = f_i^{-1}(\mathcal{M}_i)$ and $f_i^s : R_i^s \to \mathcal{M}_i$ the restriction of f_i. In this situation, there exist universal bundles $\mathcal{U}_i^{ss}$ on $R_i^{ss} \times X$. Let $\mathcal{U}_i^s$ be its restriction to $R_i^s \times X$. The group $GL(N_i)$ acts on R_i^s, with the centre acting trivially and such that $PGL(N_i)$ acts freely. The quotient of R_i^s by $PGL(N_i)$ is the moduli space of stable bundles $\mathcal{M}_i$.

We do the construction over

$$R_1^s \times \dots \times R_r^s \setminus (f_1^s \times \dots \times f_r^s)^{-1} \Delta_r.$$

For $r=1$, the result is trivial. The first non-trivial case is $r=2$, which we consider as the base case. Let $q_2^s : R_1^s \times R_2^s \times X \to R_1^s \times R_2^s$ and $p_i^s : R_1^s \times R_2^s \to R_i^s$ for $i = 1$, 2, be the projections. And let $\mathcal{H}_2^s$ be the sheaf

$$\mathcal{R}^1(q_2)_*(\mathscr{H}om((p_2 \times id_X)^* \mathcal{U}_2^s, (p_1 \times id_X)^* \mathcal{U}_1^s)), \tag{19}$$

where $\mathscr{H}om$ is the sheaf of homomorphisms. Note that $\mathrm{Hom}(\mathcal{U}_2^s|_{\{m_2\} \times X}, \mathcal{U}_1^s|_{\{m_1\} \times X}) = 0$ since both are stable bundles of the same slope, then $h^1((\mathcal{U}_2^s|_{\{m_2\} \times X})^* \otimes (\mathcal{U}_1^s|_{\{m_1\} \times X}))$ is independent of the choice of the point $(m_1, m_2) \in R_1^s \times R_2^s \setminus (f_1^s \times f_2^s)^{-1} \Delta_2$. Hence $\mathcal{H}_2^s$ is a bundle on $R_1^s \times R_2^s \setminus (f_1^s \times f_2^s)^{-1} \Delta_2$. We consider the projectivization of $\mathcal{H}_2^s$, $\mathbb{P}(\mathcal{H}_2^s)$. The centre of $GL(N_1) \times GL(N_2)$ acts trivially on the projective bundle associated to $\mathcal{H}_2^s$ and so $PGL(N_1) \times PGL(N_2)$ acts freely on $\mathbb{P}(\mathcal{H}_2^s)$. Using Kempf's descent Lemma (see [LeP2] page 138, [DN] Theorem 2.3) we obtain that $\mathbb{P}(\mathcal{H}_2^s)/PGL(N_1) \times PGL(N_2)$ is

a projective fibration over $\mathcal{M}_1 \times \mathcal{M}_2 \setminus \Delta_2$ that satisfies the properties of the proposition. This projective fibration is identified to $\mathcal{G}_{(n_1,n_2)}$. Moreover, let $\mathcal{O}_{P_2}(1)$ be the tautological bundle of the projective bundle $\mathbb{P}(\mathcal{H}_2^s)$, $\pi_{P_2}^s : \mathbb{P}(\mathcal{H}_2^s) \to \mathcal{U}_1^s \times \mathcal{U}_2^s$ and let $p_{\mathbb{P}(\mathcal{H}_2^s)} : \mathbb{P}(\mathcal{H}_2^s) \times X \to \mathbb{P}(\mathcal{H}_2^s)$ be the projection. We are in the hypotheses of Remark 2.11, so there exists a vector bundle $\mathcal{F}_2^s$ over $\mathbb{P}(\mathcal{H}_2^s) \times X$ and an exact sequence

$$0 \to (\pi_P^s \times id_X)^*(p_1^s \times id_X)^*\mathcal{U}_1^s \otimes p_{\mathbb{P}(\mathcal{H}_2^s)}^*\mathcal{O}_{P_2}(1) \to \mathcal{F}_2^s \to (\pi_P^s \times id_X)^*(p_2^s \times id_X)^*\mathcal{U}_2^s \to 0, \tag{20}$$

which is universal in the sense of the projective version of Proposition 2.10.

In the inductive step we assume that there exists a sequence of projective fibrations

$$\mathcal{G}'_{(n_1,\ldots,n_{r-1})} \to \ldots \to \mathcal{G}'_{(n_1,n_2)} \to R_1^s \times \ldots \times R_{r-1}^s \backslash (f_1^s \times \ldots \times f_{r-1}^s)^{-1}\Delta_{r-1},$$

and a universal family

$$0 = \mathcal{F}_0^s \subset \mathcal{U}_1^s \subset \ldots \subset \mathcal{F}_{r-1}^s$$

parametrized by $\mathcal{G}'_{(n_1,\ldots,n_{r-1})}$. The group $PGL(N_1) \times \ldots \times PGL(N_{r-1})$ acts freely on $\mathcal{G}'_{(n_1,\ldots,n_{r-1})}$ in such a way that there exists a quotient sequence

$$\mathcal{G}_{(n_1,\ldots,n_{r-1})} \to \ldots \to \mathcal{G}_{(n_1,n_2)} \to \mathcal{M}_1 \times \ldots \times \mathcal{M}_{r-1} \backslash \Delta_{r-1}.$$

Let us show that this is also true for $\underline{n}$. By the inductive step we have constructed a sheaf $\mathcal{H}_{r-1}^s$ as (19), over $\mathbb{P}(\mathcal{H}_{r-2}^s) \times R_{r-1}^s$. Note that $\mathcal{H}_{r-1}^s$ is actually a bundle on $\mathbb{P}(\mathcal{H}_{r-2}^s) \times R_{r-1}^s$. We consider the projectivization of $\mathcal{H}_{r-1}^s$, $\mathbb{P}(\mathcal{H}_{r-1}^s)$. One has that $PGL(N_1) \times \ldots \times PGL(N_{r-1})$ acts trivially on $\mathbb{P}(\mathcal{H}_{r-1}^s)$, and then it acts trivially on $\mathcal{G}'_{(n_1,\ldots,n_{r-1})}$, which implies the existence of a sequence of projective fibrations on the quotient. Let $\mathcal{O}_{P_{r-1}}(1)$ be the tautological bundle of the projective bundle $\mathbb{P}(\mathcal{H}_{r-1}^s)$. Let $\pi_P : \mathbb{P}(\mathcal{H}_{r-1}^s) \to \mathbb{P}(\mathcal{H}_{r-2}^s) \times R_{r-1}^s$ and let $p_{\mathbb{P}(\mathcal{H}_{r-1}^s)}^s : \mathbb{P}(\mathcal{H}_{r-1}^s) \times X \to \mathbb{P}(\mathcal{H}_{r-1}^s)$ be the projection. For the existence of $\mathcal{F}_{r-1}^s$ it is required that for every point $(m, m') \in \mathbb{P}(\mathcal{H}_{r-2}^s) \times R_{r-1}^s$ we have that $\mathrm{Hom}(\mathcal{U}_{r-1}^s|_{\{m'\}\times X}, \mathcal{F}_{r-2}^s|_{\{m\}\times X}) = 0$ (see Remark 2.11).

Consider now the projections $q_r^s : \mathbb{P}(\mathcal{H}_{r-1}^s) \times R_r^s \times X \to \mathbb{P}(\mathcal{H}_{r-1}^s) \times R_r^s$, $p : \mathbb{P}(\mathcal{H}_{r-1}^s) \times R_r^s \to \mathbb{P}(\mathcal{H}_{r-1}^s)$ and $p_r^s : \mathbb{P}(\mathcal{H}_{r-1}^s) \times R_r^s \to R_r^s$. And let $\mathcal{H}_r^s$ be the sheaf

$$\mathcal{R}^1(q_r^s)_*(\mathscr{H}om((p_r^s \times id_X)^*\mathcal{U}_r^s, (p \times id_X)^*\mathcal{F}_{r-1}^s)),$$

where $\mathscr{H}om$ is the sheaf of homomorphisms. Since $\mathrm{Hom}(\mathcal{U}_r^s|_{\{m_3\}\times X}, \mathcal{F}_{r-1}^s|_{\{m_4\}\times X}) = 0$ for all $(m_4, m_3) \in \mathbb{P}(\mathcal{H}_{r-1}^s) \times R_r^s$, one has that $\mathcal{H}_r^s$ is a bundle on $\mathbb{P}(\mathcal{H}_{r-1}^s) \times R_r^s$. Note that if there were a non-zero morphism from $\mathcal{U}_r^s|_{\{m_3\}\times X}$ to $\mathcal{F}_{r-1}^s|_{\{m_4\}\times X}$, then one could find a non-zero morphism from $\mathcal{U}_r^s|_{\{m_3\}\times X}$ to $\mathcal{U}_{r-1}^s|_{\{m''\}\times X}$ for some $m'' \in R_{r-1}^s$, but this is not possible because these are non-isomorphic stable bundles of the same slope. We consider the projectivization of $\mathcal{H}_r^s$, $\mathbb{P}(\mathcal{H}_r^s)$. One easily see that $PGL(N_1) \times \ldots \times PGL(N_r)$ acts trivially on $\mathbb{P}(\mathcal{H}_r^s)$, and then it acts trivially on $\mathcal{G}'_{\underline{n}}$, which implies the existence of the required sequence of projective fibrations on the quotient. Moreover, let $\mathcal{O}_{P_r}(1)$ be the tautological bundle of the projective bundle $\mathbb{P}(\mathcal{H}_r^s)$. Let $\pi_{P_r} : \mathbb{P}(\mathcal{H}_r^s) \to \mathbb{P}(\mathcal{H}_{r-1}^s) \times R_r^s$ and let $p_{\mathbb{P}(\mathcal{H}_r^s)} : \mathbb{P}(\mathcal{H}_r^s) \times X \to \mathbb{P}(\mathcal{H}_r^s)$ be the projection. Then we are in the hypotheses of Remark 2.11, so there exists a vector bundle $\mathcal{F}_r^s$ over $\mathbb{P}(\mathcal{H}_r^s) \times X$ and an exact sequence

$$0 \to (\pi_{P_r} \times id_X)^*(p \times id_X)^*\mathcal{F}_{r-1}^s \otimes p_{\mathbb{P}(\mathcal{H}_r^s)}^*\mathcal{O}_{P_r}(1) \to \mathcal{F}_r^s \to (\pi_{P_r} \times id_X)^*(p_r^s \times id_X)^*\mathcal{U}_r^s \to 0,$$

which is universal in the sense of the projective version of Proposition 2.10.

Finally, when $\gcd(n_i, d_i) = 1$ for all i one has that there exists Poincaré bundles $\mathcal{P}_i$ over $\mathcal{M}_i$. Then we can repeat the previous argument at the moduli space level obtaining a universal filtration

$$0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r$$

parametrized by $\mathcal{G}_{\underline{n}}$. □

Remark 6.6. The proof of the previous proposition shows that there always exists a universal filtration at the Quot-scheme level.

Proposition 6.7. *The Jordan–Hölder filtration*

$$0 \subset F_1 \subset F_2 \subset \ldots \subset F_r = F, \tag{21}$$

of F is unique if and only if no sequence

$$0 \to Q_i \to F_{i+1}/F_{i-1} \to Q_{i+1} \to 0 \tag{22}$$

for $0 < i \leqslant r-1$, splits. If no two Q_i are isomorphic, this is equivalent to saying that

$$Hom(Q_{i+1}, F_{i+1}/F_{i-1}) = 0 \tag{23}$$

for $0 < i \leqslant r-1$.

Proof. For the first statement, suppose that the Jordan–Hölder filtration (21) is not unique, then if we have two Jordan–Hölder filtrations for F there exists an index $i+1 < r$ such that $F_j = F'_j$ for all $j < i+1$ and $F_{i+1} \neq F'_{i+1}$.

If $F'_{i+1} \subsetneqq F_{i+1}$, there is a non-zero morphism of vector bundles $\psi : F'_{i+1}/F_i \to F_{i+1}/F_i$. Since F'_{i+1}/F_i and F_{i+1}/F_i are stable bundles having the same slope and $\psi \neq 0$, we get that $F'_{i+1}/F_i \cong F_{i+1}/F_i$. Hence F'_{i+1} and F_{i+1} have the same rank, so $F'_{i+1} = F_{i+1}$, which is a contradiction. Then $F'_{i+1} \nsubseteq F_{i+1}$. It follows that there exists a unique $j \geqslant i+2$ such that $F'_{i+1} \subset F_j$ but $F'_{i+1} \nsubseteq F_{j-1}$. This implies that there is a non-zero bundle morphism $F'_{i+1} \to F_j$, which induces $\varphi : F'_{i+1}/F_i \to F_j/F_{j-1} = Q_j$. Since $\varphi \neq 0$ and F'_{i+1}/F_i and $F_j/F_{j-1} = Q_j$ are stable bundles having the same slope, then φ is an isomorphism.

The bundle F_j/F_i is the middle term of the following exact sequence

$$0 \to F_{j-1}/F_i \to F_j/F_i \to Q_j \to 0. \tag{24}$$

We have that F'_{i+1}/F_i is a subbundle of F_j/F_i which is isomorphic to Q_j, which is stable. One then has that the sequence (24) splits. It follows that

$$0 \to Q_{j-1} \to F_j/F_{j-2} \to Q_j \to 0 \tag{25}$$

splits.

Suppose now that for some $1 \leqslant i \leqslant r$ the sequence

$$0 \to Q_i \to F_{i+1}/F_{i-1} \to Q_{i+1} \to 0 \tag{26}$$

splits, that is $F_{i+1}/F_{i-1} \cong Q_i \oplus Q_{i+1}$. We have a Jordan–Hölder filtration of F

$$0 \subset F_1 \subset F_2 \subset \ldots \subset F_{i-1} \subset F_i \subset F_{i+1} \subset \ldots \subset F_r = F, \tag{27}$$

then we can consider the exact sequence

$$0 \to F_{i-1} \to F_i \to Q_i \to 0. \tag{28}$$

If we take the tensor product by Q^*_{i+1} and then cohomology, we get the following exact sequence

$$H^1(Q^*_{i+1} \otimes F_{i-1}) \to H^1(Q^*_{i+1} \otimes F_i) \to H^1(Q^*_{i+1} \otimes Q_i) \to 0. \tag{29}$$

The fact that (26) is split implies that its extension class in $H^1(Q^*_{i+1} \otimes Q_i)$ is zero. From the exactness of the previous sequence, there is an extension

$$0 \to F_{i-1} \to F'_i \to Q_{i+1} \to 0 \tag{30}$$

from which the canonical extension

$$0 \to F_i \to F_{i+1} \to Q_{i+1} \to 0$$

is induced. There is also a commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & & \\
 & Q_i \arrow[u] \arrow[r, equal] & Q_i \arrow[u] & & \\
0 \arrow[r] & F_i \arrow[u] \arrow[r] & F_{i+1} \arrow[u] \arrow[r] & Q_{i+1} \arrow[r] & 0 \\
0 \arrow[r] & F_{i-1} \arrow[u] \arrow[r] & F'_i \arrow[u] \arrow[r] & Q_{i+1} \arrow[u, equal] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{31}$$

From this, one gets two different Jordan–Hölder filtrations of F

$$0 \subset F_1 \subset F_2 \subset \ldots \subset F_{i-1} \subset F_i \subset F_{i+1} \subset \ldots \subset F_r = F, \tag{32}$$

and

$$0 \subset F_1 \subset F_2 \subset \ldots \subset F_{i-1} \subset F'_i \subset F_{i+1} \subset \ldots \subset F_r = F. \tag{33}$$

This concludes the proof of the first statement.

For the second statement, if a sequence (22) splits and no two Q_i are isomorphic, then the condition $\mathrm{Hom}(Q_{i+1}, F_{i+1}/F_{i-1}) = 0$ fails. Conversely, if no two Q_i are isomorphic and there is a non-zero bundle morphism $Q_{i+1} \to F_{i+1}/F_{i-1}$, then (22) splits. □

We introduce now the following subset of $\mathcal{G}_{\underline{n}}$.

Definition 6.8. We define $\mathcal{E}_{\underline{n}}$ as the set of bundles which admit Jordan–Hölder filtrations in $\mathcal{G}_{\underline{n}}$ satisfying

$$Hom(Q_{i+1}, F_{i+1}/F_{i-1}) = 0 \tag{34}$$

for every i. Note that $\mathcal{E}_{(n_1,n_2)} \cong \mathcal{G}_{(n_1,n_2)}$.

Proposition 6.9. *$\mathcal{E}_{\underline{n}}$ has a natural structure of quasi-projective variety.*

Proof. The conditions (34) are open by the Semicontinuity Theorem, so this follows from Propositions 6.5 and 6.7. □

Now, we can calculate the number of parameters on which $\mathcal{E}_{\underline{n}}$ depends.

Lemma 6.10. *The elements of $\mathcal{E}_{\underline{n}}$ depend on exactly*

$$\dim \widetilde{\mathcal{M}}(n-k, d) - \sum_{1 \le j < i \le r} n_i n_j (g-1)$$

parameters.

Proof. We use induction on r. The case $r = 1$ is trivial. Assume now that $r \geq 2$ and that the lemma is true for Jordan–Hölder filtrations of length $r - 1$. By Definition 6.8 and Proposition 6.7, any $F \in \mathcal{E}_{\underline{n}}$ has a unique Jordan–Hölder filtration, and in particular there is a non-split extension

$$0 \to F_{r-1} \to F \to Q_r \to 0$$

uniquely determined up to a scalar multiple. Since $Q_i \not\cong Q_j$ for $i \neq j$, we have $h^0(Q_r^* \otimes F_{r-1}) = 0$, so by Riemann-Roch,

$$h^1(Q_r^* \otimes F_{r-1}) = (n - k - n_r)n_r(g-1).$$

By the inductive hypothesis, the non-split extensions depend on at most

$$\dim\widetilde{\mathcal{M}}(n-k-n_r, d-d_r) - \sum_{1\le j<i\le r-1} n_i n_j(g-1) + n_r^2(g-1) + 1 + (n-k-n_r)n_r(g-1) - 1$$

parameters. It is easy to check that this coincides with the required formula. □

It will be convenient for our descriptions to use a canonical filtration associated to our semistable vector bundle that encodes the information about the Jordan–Hölder filtrations admitted by this bundle. It turns out that for a given semistable vector bundle F, there is a canonical filtration that satisfies certain properties as it is proved in the following Lemma.

Lemma 6.11. *For every semistable vector bundle F, there is a canonical filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_s = F, \tag{35}$$

such that the quotients E_i/E_{i-1} are direct sums of stable vector bundles E' satisfying $\mu(E') = \mu(F)$ for $1 \le i \le s$ and F/E_{i-1} contains no subbundle which is the direct sum of E_i/E_{i-1} with a stable vector bundle of the same slope as F. Actually, if (15) is a Jordan–Hölder filtration of F, then $\oplus_{i=1}^s E_i/E_{i-1} \cong \oplus_{j=1}^r F_j/F_{j-1} = \mathrm{grad}F$.

Proof. We write $\mu := \mu(F)$. To prove existence of (35), we proceed by induction on s, the case $s = 1$ being trivial. There certainly exist subbundles of F which are direct sums of stable bundles of slope μ (for instance, take the first term in a Jordan–Hölder filtration, which is actually stable of slope μ). Let E_1 be a subbundle which is maximal with respect to this property; then clearly F contains no subbundle which is the direct sum of E_1 with a stable bundle of slope μ. The result now follows by applying the induction hypothesis to F/E_1.

For uniqueness, it is clearly sufficient by induction to prove uniqueness of E_1. To see this, suppose we have two different subbundles G and H which are both direct sums of stable bundles of slope μ and are both maximal with respect to this property. Consider the exact sequence

$$0 \to G\cap H \to G \to G/G\cap H \to 0.$$

Since G is semistable of slope μ, the subbundle $G\cap H$ has slope $\le \mu$ and $G/G\cap H$ has slope $\ge \mu$. Now $G/G\cap H$ is a subsheaf of F/H, which is semistable of slope μ. It follows that both inequalities are equalities and that $G\cap H$ and $G/G\cap H$ are both semistable of slope μ. Hence each of the stable direct factors of G maps to $G/G\cap H$ either by an isomorphism onto a subbundle or by 0. It follows that there exists a sum of direct factors of G which maps isomorphically to $G/G\cap H$ and the above sequence splits, i.e. $G \cong G\cap H \oplus G/G\cap H$. Moreover, by the uniqueness of direct sum decompositions, $G\cap H$ and $G/G\cap H$ are both direct sums of stables. The same argument applies to H and it follows that

$$G + H \cong G\cap H \oplus G/G\cap H \oplus H/G\cap H.$$

So $G+H$ is a direct sum of stable bundles of slope μ, contradicting the maximality of G and H. □

Later on in this paper, the use of these canonical filtrations will simplify our descriptions. From now on we restrict our study to the cases $r = 2$ and $r = 3$, cases in which we have complete descriptions.

6.1. **The case $r = 2$.** Consider the extensions

$$0 \to Q_1 \to F \to Q_2 \to 0. \tag{36}$$

We have $\mu(Q_1) = \mu(F) = \mu(Q_2)$. We denote by (n_1, d_1) and (n_2, d_2) the invariants of Q_1 and Q_2 respectively. In this case $\underline{n} = (n_1, n_2)$ is the type of (36) and $n_1+n_2 = n-k$. Note that $\mathrm{grad}(F) = Q_1\oplus Q_2$.

§. The non-split case. We will classify the non-split extensions (36) in which F_1 and Q_2 are stable bundles. As we have already seen, either $Hom(Q_2, Q_1) = 0$, or $Q_1 \cong Q_2$. If $Hom(Q_2, Q_1) = 0$, then $h^0(Q_2^* \otimes Q_1) = 0$. If $Q_1 \cong Q_2$ then $h^0(Q_2^* \otimes Q_1) = 1$. Here the quasi-projective variety $\mathcal{E}_{\underline{n}}$ (see Definition 6.8) is the space of extension classes of non-splitting extensions (36) satisfying $Hom(Q_2, Q_1) = 0$. We need the following

Definition 6.12. Let $\mathcal{E}'_{\underline{n}}$ be the space of extension classes of non-splitting extensions (36) satisfying $Q_1 \cong Q_2$.

From Proposition 6.7 and Definition 6.8, we have

Lemma 6.13. *With the above conditions, the non-splitting extension (36) is uniquely determined by F (up to scalar multiples). In particular the type $\underline{n}$ of (36) is determined by F in this case.*

We know that the extensions of Q_2 by Q_1 are classified, up to equivalence, by $H^1(Q_2^* \otimes Q_1)$. By Riemann-Roch Theorem

$$h^1(Q_2^* \otimes F_1) = n_1(n - k - n_1)(g - 1) + h^0(Q_2^* \otimes F_1). \tag{37}$$

We give a complete description of $\mathcal{E}_{\underline{n}}$ and $\mathcal{E}'_{\underline{n}}$.

Proposition 6.14. (i) *When $n_1 \neq \frac{1}{2}(n-k)$ the space $\mathcal{E}_{\underline{n}}$ is isomorphic to a projective bundle over $\mathcal{M}_1 \times \mathcal{M}_2$, with fiber the projective space of dimension $n_1(n-k-n_1)(g-1) - 1$ and $\mathcal{E}'_{\underline{n}} = \emptyset$.*

(ii) *When $n_1 = \frac{1}{2}(n-k)$:*

The space $\mathcal{E}_{\underline{n}}$ is isomorphic to a projective bundle over $\mathcal{M}_1 \times \mathcal{M}_1 \setminus \Delta$, with fiber the projective space of dimension $n_1^2(g-1) - 1$ and where $\Delta := \{(F', Q) \in \mathcal{M}_1 \times \mathcal{M}_1$ such that $F' \cong Q\}$.

The space $\mathcal{E}'_{\underline{n}}$ is isomorphic to a projective bundle over $\mathcal{M}_1$, with fiber the projective space of dimension $n_1^2(g-1)$.

Proof. The construction for $\mathcal{E}_{\underline{n}}$ in both cases appears in the proof of Proposition 6.5.

Regarding $\mathcal{E}'_{\underline{n}}$, when $n_1 \neq \frac{1}{2}(n-k)$, one has that Q_1 and Q_2 are stable bundles of the same slope and different rank, then $Q_1 \ncong Q_2$, hence $\mathcal{E}'_{\underline{n}} = \emptyset$. When $n_1 = \frac{1}{2}(n-k)$, consider first the case in which $\gcd(n_1, d_1) = 1$ and let $\mathcal{M} = \mathcal{M}_1$. Let $\mathcal{P}$ be the Poincaré bundle on $\mathcal{M} \times X$ and $H = \mathscr{H}om((p_2 \times id_X)^*\mathcal{P}, (p_1 \times id_X)^*\mathcal{P})$. Consider the following commutative diagram

$$\begin{tikzcd} \mathcal{M}' = \mathcal{M} \times X \arrow[r, "\Delta'"] \arrow[d, "q'"'] & \mathcal{M} \times \mathcal{M} \times X \arrow[d, "q"] \\ \mathcal{M} \arrow[r, "\Delta"] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

where Δ is the diagonal morphism, q and q' are the natural projections, and $\mathcal{M}'$ is the fiber product between $\mathcal{M} \times \mathcal{M} \times X$ and $\mathcal{M}$ over $\mathcal{M} \times \mathcal{M}$.

The pull-back by Δ of the sheaf $\mathcal{R}^1 q_* H$ is a locally free sheaf on $\mathcal{M}$, $\Delta^* \mathcal{R}^1 q_* H$. Using the base change formula ([H, III. §9. Proposition 9.3]) we get that

$$\Delta^* \mathcal{R}^1 q_* H \simeq \mathcal{R}^1 q'_* \Delta'^* H.$$

So the sheaf $\mathcal{R}^1 q'_* \Delta'^* H$ is a bundle on $\mathcal{M}$ that satisfies all the required properties. By (37) the projective bundle associated to $\mathcal{R}^1 q'_* \Delta'^* H$ has dimension $n_1^2(g-1)$.

When the invariants are not coprime an argument similar to the one we use in the proof of Proposition 6.5 gives us the result. □

Now, using Lemma 6.10 we can calculate the number of parameters on which $\mathcal{E}_{\underline{n}}$ and $\mathcal{E}'_{\underline{n}}$ depend.

Lemma 6.15. *The elements of $\mathcal{E}_{\underline{n}}$ depend on exactly*

$$\dim\widetilde{\mathcal{M}}(n-k,d) - n_1(n-k-n_1)(g-1)$$

parameters. When $n_1 = \frac{1}{2}(n-k)$, the elements of $\mathcal{E}'_{\underline{n}}$ depend on exactly

$$\dim\widetilde{\mathcal{M}}(n-k,d) - 2n_1^2(g-1)$$

parameters.

Proof. The first statement is deduced from Lemma 6.10. Note that this computation does not depend on either $n_1 = \frac{1}{2}(n-k)$ or $n_1 \neq \frac{1}{2}(n-k)$. For the numbers of parameters in which $\mathcal{E}'_{\underline{n}}$ depends when $n_1 = \frac{1}{2}(n-k)$, the statement follows from an argument similar to the one used in Lemma 6.10. □

§. The split case. In this case we consider the bundles $F \cong Q_1 \oplus Q_2$, such that Q_1 and Q_2 are stable bundles, and $\mu(F) = \mu(Q_1) = \mu(Q_2)$.

Definition 6.16. Let $\mathcal{SE}_{\underline{n}}$ be the space that classifies the bundles $F \cong Q_1 \oplus Q_2$ satisfying $Hom(Q_2, Q_1) = 0$. And let $\mathcal{SE}'_{\underline{n}}$ be the space of those split bundles F satisfying $Q_1 \cong Q_2$.

6.17. When $Q_1 \not\cong Q_2$ and $n_1 \neq \frac{1}{2}(n-k)$, the bundles $F \cong Q_1 \oplus Q_2$ are classified by $\mathcal{M}_1 \times \mathcal{M}_2$. When $n_1 = \frac{1}{2}(n-k)$ and $Q_1 \not\cong Q_2$, then these are classified by $(\mathcal{M}_1 \times \mathcal{M}_1 \setminus \Delta)/(\mathbb{Z}/2)$ where the group $\mathbb{Z}/2$ acts permuting the factors. Finally, when $Q_1 \cong Q_2$, the bundles are classified by $\mathcal{M}_1$.

We can again compute the number of parameters on which $\mathcal{SE}_{\underline{n}}$ and $\mathcal{SE}'_{\underline{n}}$ depend.

Lemma 6.18. *The elements of $\mathcal{SE}_{\underline{n}}$ depend on exactly*

$$\dim\widetilde{\mathcal{M}}(n_1,d_1) + \dim\widetilde{\mathcal{M}}(n-k-n_1,d-d_1)$$

parameters. When $n_1 = \frac{1}{2}(n-k)$, the elements of $\mathcal{SE}'_{\underline{n}}$ depend on exactly

$$\dim\widetilde{\mathcal{M}}(n_1,d_1)$$

parameters.

6.2. **The case $r=3$.** When $r=3$, we will classify the different possible sets of Jordan–Hölder filtrations that are admitted by our strictly semistable vector bundles.

When $r=3$, the Jordan–Hölder filtrations admitted by F are of the form

$$0 \subset F_1 \subset F_2 \subset F_3 = F. \tag{38}$$

In order to construct "universal" filtrations we must construct universal extensions as we did for $r=2$ (see Proposition 6.14) in several steps, which allow us to get universal bundles F_i. These bundles could be split bundles or nonsplit ones.

Let us fix the notation $Q_1 = F_1$ and $Q_i = F_i/F_{i-1}$, for all $i = 2, \dots r$. The bundles Q_i are stable and of the same slope as F.

Let $\underline{n} = (n_1, n_2, n_3) = (\mathrm{rank}(Q_1), \mathrm{rank}(Q_2), \mathrm{rank}(Q_3))$ be the type of F. We denote by $\underline{n}(\sigma)$ the type $(n_{\sigma(1)}, n_{\sigma(2)}, n_{\sigma(3)})$ where σ is a permutation of three elements, for example $\underline{n}(12) = (n_2, n_1, n_3)$. Assume that the Q_1, Q_2, Q_3 have the same slope. We assume further that the graded object associated to the semistable vector bundle F is $\mathrm{grad}F = Q_1 \oplus Q_2 \oplus Q_3$. We consider the following exact sequences

$$0 \to Q_1 \to F_2 \to Q_2 \to 0 \tag{39}$$

$$0 \to F_2 \to F \to Q_3 \to 0 \tag{40}$$

and

$$0 \to Q_2 \to F/Q_1 \to Q_3 \to 0. \tag{41}$$

canonically associated to the Jordan–Hölder filtration (38). Let us denote the classes of these extensions by e_1, e_2, η. When $e_i \neq 0$, $\eta \neq 0$, we write $[e_i]$ and $[\eta]$ for the corresponding element of the projective space. Now, the extension classes corresponding to these extensions are related by the following exact sequence in cohomology

$$\ldots \to \mathrm{Hom}(Q_3, Q_2) \to H^1(Q_3^* \otimes Q_1) \xrightarrow{\ i\ } H^1(Q_3^* \otimes F_2) \xrightarrow{\ p\ } H^1(Q_3^* \otimes Q_2) \to 0, \tag{42}$$

then $\eta = p(e_2)$.

In order to classify the bundles F which arise in this way, we distinguish the following cases by looking at whether the previous extensions split or do not. We introduce the following sets:

6.19. *Set 1.* In this case, the extensions (39), (40) and (41) are non-split. From Proposition 6.7 the Jordan–Hölder filtration of F is unique and the bundles F which arise are classified by 5-tuples

$$Q_1, Q_2, Q_3, [e_1], [e_2].$$

Note that in this case, the canonical filtration (see Lemma 6.11) coincides with the Jordan–Hölder filtration.

6.20. *Set 2.* Here, the extensions (39) and (40) are non-split, but (41) is split. In this case, the Jordan–Hölder filtration of F is not unique. There exists an extension

$$0 \to Q_1 \to F_{31} \to Q_3 \to 0, \tag{43}$$

we denote its extension class in $H^1(Q_3^* \otimes Q_1)$ by η', such that $i(\eta') = e_2$ (see (42)). Then, the bundles F which arise are classified by

$$Q_1, Q_2, Q_3, [e_1], [\eta'],$$

but note that $(Q_1, Q_2, Q_3, [e_1], [\eta'])$ and $(Q_1, Q_3, Q_2, [\eta'], [e_1])$ give the same F. To avoid duplication, we need to factor out by the action of $\mathbb{Z}/2$ permuting the bundles Q_2 and Q_3. The canonical filtration in this case is given by the following exact sequence

$$0 \to Q_1 \to F \to Q_2 \oplus Q_3 \to 0. \tag{44}$$

From the canonical filtration we will globalise the construction later on in this paper.

6.21. *Set 3.* In this case, the extension (39) is the only one that is split. The Jordan–Hölder filtration of F is not unique. The bundles F which arise are classified by

$$Q_1, Q_2, Q_3, [\eta], [\eta'].$$

As before, in order to avoid duplication, we need to factor out by the action of $\mathbb{Z}/2$ permuting the bundles Q_1 and Q_2. The canonical filtration in this case is given by the following exact sequence

$$0 \to Q_1 \oplus Q_2 \to F \to Q_3 \to 0. \tag{45}$$

6.22. *Set 4.* The only non-splitting extension is (39). Then, the bundle F is

$$F = F_2 \oplus Q_3. \tag{46}$$

The bundles F are classified by

$$Q_1, Q_2, Q_3, [e_1].$$

The canonical filtration in this case is

$$0 \to Q_1 \oplus Q_3 \to F_2 \oplus Q_3 \to Q_2 \to 0.$$

Note that if we interchange Q_2 and Q_3 we get that $F = F_{31} \oplus Q_2$, which corresponds to the case in which (39) and (41) split.

6.23. *Set 5.* Finally, we consider the case where all the extensions are split. Then

$$F \cong \mathrm{grad}F = Q_1 \oplus Q_2 \oplus Q_3. \tag{47}$$

So the bundles F are classified by Q_1, Q_2 and Q_3. To avoid duplication, we factor out by the action of S_3 permuting the bundles.

If we want to classify the strictly semistable vector bundles in $\widetilde{\mathcal{M}}(n-k,d)$ of type $\underline{n} = (n_1, n_2, n_3)$ that admit a Jordan–Hölder filtration (38) and such that $\mathrm{grad}(F) = Q_1 \oplus Q_2 \oplus Q_3$ we need also consider the possibility of $Q_i \cong Q_j$ for some i, j. This is accounted for in the following definition:

Definition 6.24. Let *Group 1* be the space whose elements are strictly semistable vector bundles in $\widetilde{\mathcal{M}}(n-k,d)$ of type $\underline{n} = (n_1, n_2, n_3)$ and such that $\mathrm{grad}(F) = Q_1 \oplus Q_2 \oplus Q_3$ where $Q_i \ncong Q_j$ for every i, j. Analogously, let *Group 2* be the space in which $Q_i \cong Q_j$ for two indices i and j. Finally, let *Group 3* be the space in which $Q_i \cong Q_j$ for all i and j.

As in the case $r = 2$ we want to classify in a geometric way all the possible situations that can appear. In this setup we shall not have a beautiful description of the spaces of quotients in terms of projective fibrations. We will still be able to give some universal constructions in all the cases, but in some of them only local ones, based always on the results of universal extensions we introduced in Subsection 2.2.

Definition 6.25. Let $\mathcal{S}_i^j\mathcal{E}_{\underline{n}}$ be the space whose elements are strictly semistable vector bundles in $\widetilde{\mathcal{M}}(n-k,d)$ of type $\underline{n} = (n_1, n_2, n_3)$. The index j means group j and the index i means the set i within the corresponding group. For the elements of group 2, we need to introduce a couple more indices α and β. Then $\mathcal{S}_i^2\mathcal{E}_{\underline{n}}^{\alpha\beta}$ means that in the graded objects of the elements of the set, the bundles Q_α and Q_β are isomorphic. Note that, with the notation of Definition 6.8 we have that $\mathcal{S}_1^1\mathcal{E}_{\underline{n}} \cong \mathcal{E}_{\underline{n}}$.

We are ready now to do our construction. Let $\mathcal{M}_i = \mathcal{M}(n_i, d_i)$ be the moduli space of stable bundles of rank n_i and degree d_i. Note that $n_1 + n_2 + n_3 = n - k$ and $d_1 + d_2 + d_3 = d$. We consider here the type $\underline{n} = (n_1, n_2, n_3)$ and $\underline{n}(\sigma)$ will be the type obtained from $\underline{n} = (n_1, n_2, n_3)$ after acting by an element $\sigma \in S_3$. The invariants we have fixed must satisfy $\frac{d_1}{n_1} = \frac{d_2}{n_2} = \frac{d_3}{n_3}$.

We are going to construct a "universal" Jordan–Hölder filtration over $\mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3 \times X$, such that for every point in the base, i.e. for a fixed graduation, we obtain a Jordan–Hölder filtration verifying the required properties. These "universal" filtrations will be filtrations associated to the elements of the different spaces we have defined in Definition 6.25.

§. **The case when** $n_1 \neq n_2 \neq n_3$. In this case it is not possible that $Q_i \cong Q_j$ for some pair $i \neq j$, so when n_1, n_2, n_3 are all distinct, the spaces $S_i^1\mathcal{E}_{\underline{n}}$ for $i = 1, \ldots, 5$ are the only ones that are non-empty.

The construction for $\mathcal{S}_1^1\mathcal{E}_{\underline{n}}$ has already been done in the proof of Proposition 6.5 when $r = 3$. There, the construction is done at the Quot scheme level which implies that this works for any (n_1, n_2, n_3). At the end we use descent lemmas in order to obtain the required construction at the moduli space level.

Here we do all the constructions at the moduli space level assuming the existence of Poincaré bundles. This is not true in general. Actually, when $(n_i, d_i) \neq 1$ the Poincaré bundles do not exist on $\mathcal{M}_i = \mathcal{M}(n_i, d_i)$. We do the construction at this level for simplicity. When the Poincaré bundles do not exists one may do the construction at the Quot scheme level and use descent lemmas afterwards as we did in the proof of Proposition 6.5.

$*$ *The construction for* $\mathcal{S}_3^1\mathcal{E}_{\underline{n}}$.

6.26. We have that $\mathcal{M}_i = \mathcal{M}(n_i, d_i)$ for $i = 1, 2, 3$ and assume that $n_1 < n_2$. Suppose again that there exist Poincaré bundles $\mathcal{P}_1$ and $\mathcal{P}_2$ on $\mathcal{M}_1 \times X$ and $\mathcal{M}_2 \times X$ respectively. Consider also the projections $p_i : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_i$ for $i = 1, 2$. There exists a universal vector bundle $(p_1 \times id_X)^*\mathcal{P}_1 \oplus (p_2 \times id_X)^*\mathcal{P}_2$ over $\mathcal{M}_1 \times \mathcal{M}_2$ in the usual sense. Let $\mathcal{P}_3$ be the Poincaré bundle on $\mathcal{M}_3 \times X$. The rest of the construction is similar to the one in the proof of Proposition 6.5. Let $q' : (\mathcal{M}_1 \times \mathcal{M}_2) \times \mathcal{M}_3 \times X \to (\mathcal{M}_1 \times \mathcal{M}_2) \times \mathcal{M}_3$. Let $\mathcal{H}'$ be the sheaf

$$\mathcal{R}^1 q'_*(\mathscr{H}om((p_3 \times id_X)^*\mathcal{P}_3, (p_1 \times id_X)^*\mathcal{P}_1 \oplus (p_2 \times id_X)^*\mathcal{P}_2)),$$

this is also a bundle on $(\mathcal{M}_1 \times \mathcal{M}_2) \times \mathcal{M}_3$. We consider the projectivization of $\mathcal{H}'$, $\mathbb{P}(\mathcal{H}')$. Let $\mathcal{O}_{P'}(1)$ be the tautological bundle of the projective bundle $\mathbb{P}(\mathcal{H}')$. For all point $(m_1, m_2, m_3) \in (\mathcal{M}_1 \times \mathcal{M}_2) \times \mathcal{M}_3$, and for all $m' \in \mathcal{H}'_{(m_1,m_2,m_3)}$, let $\mathcal{O}_{P'}(1)_{m'} = m'^*$. Let $\pi_{P'} : \mathbb{P}(\mathcal{H}') \to \mathbb{P}(\mathcal{H}) \times \mathcal{M}_3$ and let $p_{\mathbb{P}(\mathcal{H}')} : \mathbb{P}(\mathcal{H}') \times X \to \mathbb{P}(\mathcal{H}')$ be the projection.

Now, we want to construct a vector bundle $\mathcal{F}$ over $\mathbb{P}(\mathcal{H}') \times X$ satisfying all the required properties. As above, we are in the hypotheses of Remark 2.11, so there exists a vector bundle $\mathcal{F}$ over $\mathbb{P}(\mathcal{H}') \times X$ and an exact sequence

$$\begin{aligned} 0 \to (\pi_{P'} \times id_X)^*\big((p_1 \times id_X)^*\mathcal{P}_1 \oplus (p_2 \times id_X)^*\mathcal{P}_2\big) \otimes p^*_{\mathbb{P}(\mathcal{H}')}\mathcal{O}_{P'}(1) \to \mathcal{F} \to \\ \to (\pi_{P'} \times id_X)^*(p_3 \times id_X)^*\mathcal{P}_3 \to 0, \end{aligned} \tag{48}$$

such that for all $(m_1, m_2, m_3) \in (\mathcal{M}_1 \times \mathcal{M}_2) \times \mathcal{M}_3$, and for all $m' \in \mathcal{H}'_{(m_1,m_2,m_3)}$, its restriction to $\{m'\} \times X$ is the extension

$$0 \to (\mathcal{P}_{1_{m_1}} \oplus \mathcal{P}_{2_{m_2}}) \otimes m'^* \to \mathcal{F}_{m'} \to \mathcal{P}_{3_{m_3}} \to 0.$$

As a result of this construction we have obtained an extension (48) that is the globalising version of the canonical filtration (45). From this extension we will describe geometrically the corresponding stratum at the moduli space of coherent systems.

As in the above case, we must take into account the cases in which the Poincaré bundles do not exist.

Remark 6.27. The construction for $\mathcal{S}_2^1\mathcal{E}_{\underline{n}}$ is obtained by dualising $\mathcal{S}_3^1\mathcal{E}_{\underline{n}}$, while $\mathcal{S}_4^1\mathcal{E}_{\underline{n}}$ is simply $\mathcal{E}_{(n_1,n_2)} \times \mathcal{M}_3$. Finally, the construction for $\mathcal{S}_5^1\mathcal{E}_{\underline{n}}$ is given by $\mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3$.

§. The case when $n_1 = n_2 \neq n_3$. For the cases in which the 3-tuple of elements that form the graduations associated to our semistable vector bundles are elements in $\mathcal{M}_1 \times \mathcal{M}_1 \times \mathcal{M}_3 \backslash \Delta_{12}$ where Δ_{12} the diagonal in the two first components, the constructions we have described for $\mathcal{S}_i^1\mathcal{E}_{\underline{n}}$ for $i = 1, \ldots, 5$ when $n_1 \neq n_2 \neq n_3$ are the same for $n_1 = n_2 \neq n_3$.

Under the relations between the ranks of the quotient bundles we also have that $\mathcal{S}_i^3\mathcal{E}_{\underline{n}}$ are empty.

For the remaining cases, those in which the graduation is an element of $\Delta_{12} \times \mathcal{M}_3$, a more detailed study is needed. We describe here the construction of $\mathcal{S}_1^2\mathcal{E}_{\underline{n}}^{12}$, the rest of the cases come easily from a suitable combination of the following construction and the previous ones.

$*$ *The construction for* $\mathcal{S}_1^2\mathcal{E}_{\underline{n}}^{12}$. We want to construct a sort of universal Jordan–Hölder filtration over $\Delta_{12} \times \mathcal{M}_3$, where Δ_{12} is the diagonal for the two first components, note that in this case $\mathcal{M}_1 = \mathcal{M}_2$.

At the very beginning we restrict ourselves again to a hypothetical case in which we have Poincaré bundles over our moduli spaces. In spite of the fact that in general this is not true, we will be able again to work at the Quot scheme level and afterwards using descent lemmas we will be able to apply our results at the moduli of stable vector bundles level.

6.28. As in our original construction (proof of Proposition 6.5), first of all we need to construct a "universal" extension over Δ_{12}. But in this case, a universal extension in the usual sense ([NR], [R] & [S]) does not exist. This non-existence could be proved bearing in mind that Proposition 2.10 is a special case of Proposition 2.7, more precisely, the case when $\mathscr{E}xt^0_f(\mathscr{F},\mathscr{G}) = 0$ and $\mathscr{E}xt^1_f(\mathscr{F},\mathscr{G})$ commutes with base change, and the same for the projective analogues. Here we follow the notation of Proposition 6.14 (ii), and let $\mathcal{M}_1 = \mathcal{M}$ which is a reduced variety. For the morphism $q' : \mathcal{M} \times X \to \mathcal{M}$, we have that

$$\mathscr{E}xt^0_{q'}(\Delta'^*(p_2 \times id_X)^*\mathcal{P}, \Delta'^*(p_1 \times id_X)^*\mathcal{P}) \cong$$
$$\cong R^0 q'_* \Delta'^* \mathscr{H}om((p_2 \times id_X)^*\mathcal{P}, (p_1 \times id_X)^*\mathcal{P}),$$

which is not zero. So there is not a universal extension in the usual sense.

Theorem 6.29. *A "universal" family of extensions in the sense of 2.8 exists over* Δ_{12}*.*

Proof. Consider first the following commutative diagram

$$\begin{tikzcd} P \times X \arrow[r, "q'_P"] \arrow[d, "p'_P"'] & \mathcal{M} \times X \arrow[d, "q'"] \\ P \arrow[r, "g"] & \mathcal{M} \end{tikzcd}$$

where $P = \mathbb{P}(\mathscr{E}xt^1_{q'}(\Delta'^*(p_2 \times id_X)^*\mathcal{P}, \Delta'^*(p_1 \times id_X)^*\mathcal{P})^*)$ and $P \times X$ is the fiber product between P and $\mathcal{M} \times X$ over $\mathcal{M}$.

The existence of this "universal" family is based mainly on the fact that for every $m \in \mathcal{M}$, the base change morphism

$$\varphi^1(m) : R^1 q'_* \Delta'^* \mathscr{H}om((p_2 \times id_X)^*\mathcal{P}, (p_1 \times id_X)^*\mathcal{P}) \otimes k(m) \to$$
$$\to H^1(X_m, \Delta'^* \mathscr{H}om((p_2 \times id_X)^*\mathcal{P}, (p_1 \times id_X)^*\mathcal{P})_m)$$

is surjective. To see this surjectivity it is enough to note that the fibres of q' are projective curves. Then, using the Grauert theorem and the "Cohomology and base change" theorem ([H, III. §12. Corollary 12.9 and Theorem 12.11]) we conclude.

Combining the surjectivity of $\varphi^1(m)$ and the "Cohomology and Base Change" theorem we have that $\varphi^i(m)$ are isomorphisms for $i = 0$, 1. So, because $\mathcal{M}$ is reduced, we can apply Proposition 2.9. Then, there exists a family $(e_p)_{p \in P}$ of extensions of ${q'_P}^*(p_1 \times id_X)^*\mathcal{P}$ by ${q'_P}^*(p_2 \times id_X)^*\mathcal{P} \otimes {p'_P}^*\mathcal{O}_P(1)$ over $P = \mathbb{P}(\mathscr{E}xt^1_{q'}(\Delta'^*(p_2 \times id_X)^*\mathcal{P}, \Delta'^*(p_1 \times id_X)^*\mathcal{P})^*)$ which is universal, in the sense of 2.8, in the category of reduced noetherian Y-schemes for the classes of families of non-splitting extensions of ${q'_P}^*(p_1 \times id_X)^*\mathcal{P}$ by ${q'_P}^*(p_2 \times id_X)^*\mathcal{P} \otimes {p'_P}^*\mathscr{L}$ over S with arbitrary $\mathscr{L} \in Pic(S)$ modulo the canonical operation of $H^0(S, \mathcal{O}^*_S)$. □

Once we have constructed a family of extensions, $(e_p)_{p \in P}$, in the first step, for the second we use a similar argument as in the previous "universal" constructions and produce a universal extension in the usual meaning for each element of the family $(e_p)_{p \in P}$. Hence, we fix an element of $(e_p)_{p \in P}$, say

$$0 \to {q'_P}^*(p_2 \times id_X)^*\mathcal{P} \otimes {p'_P}^*\mathcal{O}_P(1) \to \mathcal{F}_p \to {q'_P}^*(p_1 \times id_X)^*\mathcal{P} \to 0.$$

Now, as in the usual notation, let $q'' : \{p\} \times \mathcal{M}_3 \times X \to \{p\} \times \mathcal{M}_3$ and $q_1 : \{p\} \times \mathcal{M}_3 \to \{p\}$, $p_3 : \{p\} \times \mathcal{M}_3 \to \mathcal{M}_3$. We have that

$$\mathcal{H}_p = \mathcal{R}^1 q''_*(\mathscr{H}om((p_3 \times id_X)^*\mathcal{P}_3, (q_1 \times id_X)^*\mathcal{F}_p))$$

is a bundle on $\{p\} \times \mathcal{M}_3$, and we consider $\mathbb{P}(\mathcal{H}_p)$. To conclude we need to construct a universal extension on $\mathbb{P}(\mathcal{H}_p) \times X$. This follows from the fact that

$$Hom(\mathcal{P}_3|_{\{m_3\}\times X}, \mathcal{F}_p) = 0$$

for all $m_3 \in \mathcal{M}_3$, note that in case there exists such a morphism, we would have another one from $\mathcal{P}_3|_{\{m_3\}\times X}$ to $\mathcal{P}|_{\{m_1\}\times X}$ for some $m_1 \in \mathcal{M}$, but this contradicts the hypotheses. Under this property, the conditions of Proposition 2.10 are fulfilled (see Remark 2.11) so we have a universal extension in the usual sense.

Remark 6.30. The constructions for the case when $n_1 = n_2 = n_3$ are analogous to the ones we have described earlier.

Remark 6.31. Regarding the number of parameters on which our sets depend, from Lemma 6.10 one obtains that $\mathcal{S}_1^1\mathcal{E}_{\underline{n}}$ for $\underline{n} = (n_1, n_2, n_3)$ depends on exactly

$$\dim \widetilde{\mathcal{M}}(n-k, d) - n_1 n_2(g-1) - n_3(n_1 + n_2)(g-1).$$

Now, to compute the number of parameters on which the elements of $\mathcal{S}_3^1\mathcal{E}_{\underline{n}}$ depend, it is enough to look at the extensions of the form

$$0 \to Q_1 \oplus Q_2 \to F \to Q_3 \to 0.$$

These extensions depend on exactly

$$\dim \widetilde{\mathcal{M}}(n-k, d) - (n_1 + n_2)n_3(g-1) - 2n_1 n_2(g-1) + 1.$$

Finally, $\mathcal{S}_1^2\mathcal{E}_{\underline{n}}^{12}$ depends on exactly

$$\dim \widetilde{\mathcal{M}}(n-k, d) - 3\dim \widetilde{\mathcal{M}}(n_1, d_1) - 2n_1 n_3(g-1) + 2.$$

The computations for the remaining cases follow in a similar fashion.

7. A stratification of $G_L(n, d, k)$

7.1. **Defining the stratification.** In this section we will define a stratification of the moduli space $G_L(n, d, k)$ when $k < n$. To this end we use the type that was defined earlier, and all the sets that we have described from a geometric point of view in the previous section. The idea is to define the different strata by looking at the quotient bundle of the BGN extension associated to every coherent system in $G_L(n, d, k)$.

By Proposition 4.5 we know that if the quotient bundle is stable, the BGN extension gives rise to an α-stable coherent system. If the quotient bundle is only strictly semistable, the BGN extension could give rise either to an α-stable or a non-α-stable coherent system.

In the previous section we studied the sets that classify the possible Jordan–Hölder filtrations that are admitted by a given semistable bundle. We define different sets in terms of all the possible splittings that can appear. These sets will be fundamental to define strata in the moduli space $G_L(n, d, k)$.

We look at the quotient bundle associated to our coherent system. The strata are defined accordingly:

Definition 7.1 (The strata)**.**

(a) Using the notation of the previous sections, for the case $r = 2$ let $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$ be the space whose elements are those $(E, V) \in G_L(n, d, k)$ such that if

$$0 \to \mathcal{O}^{\oplus k} \to E \to F \to 0$$

is the extension that represents the BGN extension class associated to (E,V) (Proposition 4.3), then the quotient bundle F is strictly semistable, has type $\underline{n}$ and is an element of $\mathcal{E}_{\underline{n}}$. We have analogous definitions when we substitute $\mathcal{E}_{\underline{n}}$ by $\mathcal{SE}_{\underline{n}}$, $\mathcal{E}'_{\underline{n}}$ and $\mathcal{SE}'_{\underline{n}}$, respectively.

(b) For the case $r=3$ we have analogous definitions for the sets we introduced in Definition 6.25.

(c) Let $\mathscr{W}^1 = G_L(n,d,k)\setminus W$ where W denotes the subvariety of $G_L(n,d,k)$ consisting of coherent systems for which the quotient bundle F is strictly semistable.

Theorem 7.2. *The sets defined in the previous definition are locally closed. Moreover, $\mathscr{W}^1$ is an open set.*

Proof. Consider $\widetilde{\mathcal{M}}_i = \widetilde{\mathcal{M}}(n_i,d_i)$ the moduli space of semistable vector bundles of rank n_i and degree d_i for $i=1,\dots,r$. Let $\mathcal{Q}_i$ be the corresponding Quot schemes, and R_i^{ss} the open set of $\mathcal{Q}_i$ of semistable points. If f_i is the morphism from R_i^{ss} to $\widetilde{\mathcal{M}}_i$, then $(\widetilde{\mathcal{M}}_i, f_i)$ is a good quotient of R_i^{ss}. Consider also the morphism

$$(f_1,\dots,f_r) : R_1^{ss}\times\ldots\times R_r^{ss} \longrightarrow \widetilde{\mathcal{M}}_1\times\ldots\times\widetilde{\mathcal{M}}_r,$$

so we have a morphism

$$R_1^{ss}\times\ldots\times R_r^{ss} \longrightarrow \widetilde{\mathcal{M}}(n-k,d),$$

that factorizes through $(f_1,\dots,f_r)$, and the morphism

$$\phi_r : \widetilde{\mathcal{M}}_1\times\ldots\times\widetilde{\mathcal{M}}_r \longrightarrow \widetilde{\mathcal{M}}(n-k,d)$$

is the one sending $([E_1],\dots,[E_r])$ to $[E_1\oplus\ldots\oplus E_r]$, where the brackets mean the S-equivalence classes of vector bundles. It is easy to see, following [S] page 35, that if $(e_1,\dots,e_r)$ and $(e_1',\dots,e_r')$ are two points in $R_1^{ss}\times\ldots\times R_r^{ss}$ representing the r-tuples of vector bundles $(E_1,\dots,E_r)$ and $(E_1',\dots,E_r')$, then $\phi_r\circ(f_1,\dots,f_r)(e_1,\dots,e_r) = \phi_r\circ(f_1,\dots,f_r)(e_1',\dots,e_r')$ if and only if $\mathrm{grad}(E_1\oplus\ldots\oplus E_r)\cong\mathrm{grad}(E_1'\oplus\ldots\oplus E_r')$.

It is known that $\mathcal{M}_i$, the moduli space of stable vector bundles, is an open subset of $\widetilde{\mathcal{M}}_i$. If we consider the complement of $\mathcal{M}_1\times\ldots\times\mathcal{M}_r$ in $\widetilde{\mathcal{M}}_1\times\ldots\times\widetilde{\mathcal{M}}_r$, $(\mathcal{M}_1\times\ldots\times\mathcal{M}_r)^c$, this is a closed subset of $\widetilde{\mathcal{M}}_1\times\ldots\times\widetilde{\mathcal{M}}_r$. Because $\widetilde{\mathcal{M}}_1\times\ldots\times\widetilde{\mathcal{M}}_r$ is projective, $\phi_r(\widetilde{\mathcal{M}}_1\times\ldots\times\widetilde{\mathcal{M}}_r)$ is closed in $\widetilde{\mathcal{M}}(n-k,d)$, and $\phi_r((\mathcal{M}_1\times\ldots\times\mathcal{M}_r)^c)$ is a closed subset of $\phi_r(\widetilde{\mathcal{M}}_1\times\ldots\times\widetilde{\mathcal{M}}_r)$, so $\phi_r((\mathcal{M}_1\times\ldots\times\mathcal{M}_r)^c)^c$ is open, then $\mathcal{C}_{1\ldots r} := \phi_r((\mathcal{M}_1\times\ldots\times\mathcal{M}_r)^c)^c\cap\phi_r(\widetilde{\mathcal{M}}_1\times\ldots\times\widetilde{\mathcal{M}}_r)$ is locally closed in $\widetilde{\mathcal{M}}(n-k,d)$. The elements of $\mathcal{C}_{1\ldots r}$ are the S-equivalence classes of semistable vector bundles of rank $n-k$ and degree d, which have a Jordan–Hölder filtration whose type is $\underline{n}(\sigma) = (n_{\sigma(1)},\dots,n_{\sigma(r)})$ for some $\sigma\in S_r$.

Consider now the morphism

$$g : G_L(n,d,k) \longrightarrow \widetilde{\mathcal{M}}(n-k,d)$$

which sends every coherent system (E,V) to the S-equivalence class of the semistable quotient bundle F of the corresponding BGN extension. We have that $g^*(\phi_r(\widetilde{\mathcal{M}}_1\times\ldots\times\widetilde{\mathcal{M}}_r))$ is closed in $G_L(n,d,k)$, and $g^*\mathcal{C}_{1\ldots r}$ is open in $g^*(\phi_r(\widetilde{\mathcal{M}}_1\times\ldots\times\widetilde{\mathcal{M}}_r))$ so is locally closed. The elements of $g^*\mathcal{C}_{1\ldots r}$ are the coherent systems (E,V) of $G_L(n,d,k)$ for which $\mathrm{grad}F = E_1\oplus\ldots\oplus E_r$ with $E_i\in\mathcal{M}_i$.

(a) *The case* $r=2$. The case in which $n_1\neq\frac{1}{2}(n-k)$: Consider $R_1^{ss}\times G_L(n,d,k)$, let $\{V(t)\}_t$ and $\{W(s)\}_s$ be families of vector bundles parametrised by R_1^{ss} and $G_L(n,d,k)$ respectively. Let $\mathcal{A} := \{(t,s)\in R_1^{ss}\times G_L(n,d,k)$ such that $Hom(V(t),W(s))\neq 0\}$. Now, $\mathcal{A}$ is a closed invariant subset of $R_1^{ss}\times G_L(n,d,k)$ under the action of $PGL(N_1)\times\{id\}$. Because $(\widetilde{\mathcal{M}}_1,f_1)$ is a good quotient of R_1^{ss}, we have that $(f_1\times id_{G_L})(\mathcal{A})$ is closed in $\widetilde{\mathcal{M}}_1\times G_L(n,d,k)$ (see [N2]), hence its image in $G_L(n,d,k)$ is also closed. We call this image $\mathcal{I}_{12}$. Analogously, let $\mathcal{A}' := \{(t,s)\in R_1^{ss}\times G_L(n,d,k)$ such that $Hom(W(s),V(t))\neq 0\}$ and denote by $\mathcal{I}'_{12}$ to the image in $G_L(n,d,k)$ of $(f_1\times id_{G_L})(\mathcal{A}')$ which is also closed.

We have that $\mathcal{I}_{12} \cap g^*\mathcal{C}_{12} = \mathscr{W}_{\mathcal{E}_{\underline{n}}} \cup \mathscr{W}_{\mathcal{S}\mathcal{E}_{\underline{n}}}$ and $\mathcal{I}'_{12} \cap g^*\mathcal{C}_{12} = \mathscr{W}_{\mathcal{E}_{\underline{n}(12)}} \cup \mathscr{W}_{\mathcal{S}\mathcal{E}_{\underline{n}}}$. Then, $\mathscr{W}_{\mathcal{S}\mathcal{E}_{\underline{n}}} = \mathcal{I}_{12} \cap \mathcal{I}'_{12} \cap g^*\mathcal{C}_{12}$, $\mathscr{W}_{\mathcal{E}_{\underline{n}}} = (\mathcal{I}_{12} \setminus \mathcal{I}_{12} \cap \mathcal{I}'_{12}) \cap g^*\mathcal{C}_{12}$ and $\mathscr{W}_{\mathcal{E}_{\underline{n}(12)}} = (\mathcal{I}'_{12} \setminus \mathcal{I}_{12} \cap \mathcal{I}'_{12}) \cap g^*\mathcal{C}_{12}$ which are all locally closed.

The case in which $n_1 = \frac{1}{2}(n-k)$: Here we have two different possibilities, the points in the diagonal and the ones out of the diagonal. Consider first the diagonal $\Delta := \{(F', Q) \in \widetilde{\mathcal{M}}_1 \times \widetilde{\mathcal{M}}_1$ such that $F' \cong Q\}$ which is a closed set. Then $\phi_2(\Delta)$ is also closed in $\widetilde{\mathcal{M}}(n-k, d)$ and $g^*(\phi_2(\Delta) \cap \mathcal{C}_{11})$ is closed in $G_L(n,d,k)$. Define $\mathcal{J}_{11} := \{s \in G_L(n,d,k) : h^0(End(W(s))) = 4\}$. By semi-continuity theorem ([H] page 288), the set $\mathcal{J}_{11}$ is closed in $G_L(n,d,k)$. So $\mathscr{W}_{\mathcal{S}\mathcal{E}'_{\underline{n}}} = \mathcal{J}_{11} \cap g^*(\phi_2(\Delta) \cap \mathcal{C}_{11})$ and $\mathscr{W}_{\mathcal{E}'_{\underline{n}}} = g^*(\phi_2(\Delta) \cap \mathcal{C}_{11}) \setminus \mathcal{J}_{11} \cap g^*(\phi_2(\Delta) \cap \mathcal{C}_{11})$ are both locally closed.

Secondly, for the points off the diagonal, we define $\mathcal{I}_{12}$ as in the first part of (a). One has that $\mathcal{I}_{12} \cap g^*(\phi_2(\Delta))^c \cap g^*(\mathcal{C}_{11}) = \mathscr{W}_{\mathcal{E}'_{\underline{n}}} \cup \mathscr{W}_{\mathcal{S}\mathcal{E}'_{\underline{n}}}$. Consider now $\mathcal{J} := \{s \in G_L(n,d,k) : h^0(End(W(s))) = 2\}$. This set is closed in $G_L(n,d,k)$, and $\mathscr{W}_{\mathcal{S}\mathcal{E}'_{\underline{n}}} = \mathcal{J} \cap \mathcal{I}_{12} \cap g^*(\phi_2(\Delta))^c \cap g^*(\mathcal{C}_{11})$ and $\mathscr{W}_{\mathcal{E}'_{\underline{n}}} = \mathcal{I}_{12} \cap g^*(\phi_2(\Delta))^c \cap g^*(\mathcal{C}_{11}) \setminus \mathcal{J} \cap \mathcal{I}_{12} \cap g^*(\phi_2(\Delta))^c \cap g^*(\mathcal{C}_{11})$ which are both locally closed. So we conclude.

(b) *The case* $r = 3$. As we have seen before, several cases, depending on the relations between the different ranks, should be considered.

The case in which $n_1 \neq n_2 \neq n_3$: Recall that $\widetilde{\mathcal{M}}_i = \widetilde{\mathcal{M}}(n_i, d_i)$ the moduli space of semistable vector bundles of rank n_i and degree d_i for $i = 1, \dots, r$. Let $\mathcal{Q}_i$ be the corresponding Quot schemes, and R_i^{ss} the open set of $\mathcal{Q}_i$ of semistable points. If f_i is the morphism from R_i^{ss} to $\widetilde{\mathcal{M}}_i$, then $(\widetilde{\mathcal{M}}_i, f_i)$ is a good quotient of R_i^{ss}. Let $\widetilde{\mathcal{M}}_{ij} = \widetilde{\mathcal{M}}(n_i + n_j, d_i + d_j)$ the moduli space of semistable vector bundles of rank $n_i + n_j$ and degree $d_i + d_j$, and R_{ij}^{ss} the set of semistable points at the corresponding Quot scheme. Let $(\widetilde{\mathcal{M}}_{ij}, f_{ij})$ be the good quotient of R_{ij}^{ss}. Let $\{V_i(t_i)\}_{t_i}$, $\{V_{ij}(t_{ij})\}_{t_{ij}}$ and $\{W(s)\}_s$ be families of vector bundles parametrised by R_i^{ss}, R_{ij}^{ss} and $G_L(n,d,k)$ respectively. Let

$$\mathcal{A} := \{(t_1, t_{12}, s) \in R_1^{ss} \times R_{12}^{ss} \times G_L(n,d,k) : Hom(V_1(t_1), V_{12}(t_{12})) \neq 0, Hom(V_{12}(t_{12}), W(s)) \neq 0\}$$

Now, $\mathcal{A}$ is a closed invariant subset of $R_1^{ss} \times R_{12}^{ss} \times G_L(n,d,k)$ under the action of $PGL(N_1) \times PGL(N_{12}) \times \{id\}$. We have that $(f_1 \times f_{12} \times id_{G_L})(\mathcal{A})$ is closed in $\widetilde{\mathcal{M}}_1 \times \widetilde{\mathcal{M}}_{12} \times G_L(n,d,k)$, hence its image in $G_L(n,d,k)$ is also closed. We call this image $\mathcal{I}$. One has that $\mathcal{I} \cap g^*\mathcal{C}_{123}$ is locally closed and is identified with $\cup_{i=1}^{5} \mathscr{W}_{\mathcal{S}_i^1 \mathcal{E}_{\underline{n}}}$.

Let

$$\mathcal{A}_{ij} := \{(t_i, t_j, s) \in R_i^{ss} \times R_j^{ss} \times G_L(n,d,k) : Hom(V_i(t_i), W(s)) \neq 0, Hom(V_j(t_j), W(s)) \neq 0\}$$

and

$$\mathcal{A}'_{ij} := \{(t_i, t_j, s) \in R_i^{ss} \times R_j^{ss} \times G_L(n,d,k) : Hom(W(s), V_i(t_i)) \neq 0, Hom(W(s), V_j(t_j)) \neq 0\}$$

where $\{i,j\} \subset \{1,2,3\}$. Now, $\mathcal{A}_{ij}$ and $\mathcal{A}'_{ij}$ are closed invariant subset of $R_i^{ss} \times R_j^{ss} \times G_L(n,d,k)$ under the action of $PGL(N_i) \times PGL(N_j) \times \{id\}$. We have that $(f_i \times f_j \times id_{G_L})(\mathcal{A}_{ij})$ is closed in $\mathcal{M}_i \times \mathcal{M}_j \times G_L(n,d,k)$, hence its image in $G_L(n,d,k)$ is also closed. The same is true for $(f_i \times f_j \times id_{G_L})(\mathcal{A}'_{ij})$. We call these images $\mathcal{I}_{ij}$ and $\mathcal{I}'_{ij}$ respectively. The varieties $\mathcal{I}_{ij} \cap g^*\mathcal{C}_{123}$ and $\mathcal{I}'_{ij} \cap g^*\mathcal{C}_{123}$ are locally closed. These varieties are identified with unions of some of the strata we are studying. Among these identifications,

we have the following ones:

$$\begin{aligned}
\mathcal{I}_{12}\cap g^*\mathcal{C}_{123} &= \mathscr{W}_{\mathcal{S}^1_3\mathcal{E}_{\underline{n}}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}(23)}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}(123)}}\cup\mathscr{W}_{\mathcal{S}^1_5\mathcal{E}_{\underline{n}}}\\
\mathcal{I}_{13}\cap g^*\mathcal{C}_{123} &= \mathscr{W}_{\mathcal{S}^1_3\mathcal{E}_{\underline{n}(23)}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}(13)}}\cup\mathscr{W}_{\mathcal{S}^1_5\mathcal{E}_{\underline{n}}}\\
\mathcal{I}_{23}\cap g^*\mathcal{C}_{123} &= \mathscr{W}_{\mathcal{S}^1_3\mathcal{E}_{\underline{n}(13)}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}(12)}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}(132)}}\cup\mathscr{W}_{\mathcal{S}^1_5\mathcal{E}_{\underline{n}}}\\
\mathcal{I}'_{12}\cap g^*\mathcal{C}_{123} &= \mathscr{W}_{\mathcal{S}^1_2\mathcal{E}_{\underline{n}(13)}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}(13)}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}(132)}}\cup\mathscr{W}_{\mathcal{S}^1_5\mathcal{E}_{\underline{n}}}\\
\mathcal{I}'_{13}\cap g^*\mathcal{C}_{123} &= \mathscr{W}_{\mathcal{S}^1_2\mathcal{E}_{\underline{n}(12)}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}(12)}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}(123)}}\cup\mathscr{W}_{\mathcal{S}^1_5\mathcal{E}_{\underline{n}}}\\
\mathcal{I}'_{23}\cap g^*\mathcal{C}_{123} &= \mathscr{W}_{\mathcal{S}^1_2\mathcal{E}_{\underline{n}}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}}}\cup\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}(23)}}\cup\mathscr{W}_{\mathcal{S}^1_5\mathcal{E}_{\underline{n}}}
\end{aligned}$$

Then, one gets that $\mathscr{W}_{\mathcal{S}^1_3\mathcal{E}_{\underline{n}}}=(\mathcal{I}_{12}\cap(\mathcal{I}\setminus(\mathcal{I}\cap\mathcal{I}'_{23})))\cap g^*\mathcal{C}_{123}$, $\mathscr{W}_{\mathcal{S}^1_4\mathcal{E}_{\underline{n}}}=(\mathcal{I}_{13}\cap\mathcal{I}'_{23}\cap(\mathcal{I}\setminus(\mathcal{I}\cap\mathcal{I}_{12})))\cap g^*\mathcal{C}_{123}$ and $\mathscr{W}_{\mathcal{S}^1_1\mathcal{E}_{\underline{n}}}=(\mathcal{I}\setminus(\mathcal{I}\cap(\mathcal{I}_{12}\cup\mathcal{I}'_{23})))\cap g^*\mathcal{C}_{123}$ which are all locally closed. The remaining cases come in the same fashion.

The cases in which $n_1=n_2\neq n_3$ and $n_1=n_2=n_3$ are deduced using a suitable combination of the arguments used in (a) and (b). So we conclude.

(c) To prove that $\mathscr{W}^1$ is open it is enough to note that $\mathscr{W}^1=g^*(\mathcal{M}(n-k,d))$. □

In [BGMMN], Bradlow *et al.* find a lower bound for the codimension of $G_L(n,d,k)\setminus\mathscr{W}^1$ in $G_L(n,d,k)$. This is the following:

Lemma 7.3 ([BGMMN], Corollary 7.10). *Let $0<k<n$ and suppose that $G_L(n,d,k)\neq\emptyset$. Then the codimension of $G_L(n,d,k)\setminus\mathscr{W}^1$ in $G_L(n,d,k)$ is at least*

$$\min\{(\sum_{i<j}n_in_j)(g-1)\},\tag{49}$$

where the minimum is taken over all sequences of positive integers $r,n_1,\dots,n_r$ such that $r\geq 2$ and $\sum n_i=n-k$.

This bound is improved in the following proposition.

Proposition 7.4. *Let $0<k<n$ and suppose that $G_L(n,d,k)\neq\emptyset$. When $(n-k,d)=p\geq 2$ the codimension of $G_L(n,d,k)\setminus\mathscr{W}^1$ in $G_L(n,d,k)$ is at least*

$$\frac{p-1}{p^2}(n-k)^2(g-1).$$

Proof. In the previous lemma one needs only consider the sequences $n_1,\dots,n_r$ for which there exist d_i with $\sum_i d_i=d$ such that $\frac{d_i}{n_i}=\frac{d}{n-k}$ for all i. This means that each n_i must be a multiple of $\frac{n-k}{p}$. Given this, the minimum of (49) is attained when $r=2$ and $n_1=\frac{n-k}{p}$ and $n_2=\frac{(p-1)(n-k)}{p}$ such that $d_1=\frac{d}{p}$ and $d_2=\frac{(p-1)d}{p}$. Then

$$\min\{(\sum_{i<j}n_in_j)(g-1)\}=\frac{(p-1)}{p^2}(n-k)^2(g-1).$$

Hence we conclude. □

7.2. **Explicit description of the strata for $r=2$.** In this section, we will describe our strata for $r=2$ as complements of determinantal varieties. As above, the problem is that in general universal bundles do not exist on our moduli spaces of stable bundles. Actually, they only exist when the invariants are coprime to each other. In order to solve this problem, we will work again at the Quot scheme level -because in these schemes we have universal families of vector bundles- and afterwards we carry our construction to the moduli spaces of coherent systems via descent lemmas. In this case, we assume that the type is $\underline{n}=(n_1,n-k-n_1)$. We can consider two different subcases:

7.2.1. *The case when $n_1 \neq \frac{1}{2}(n-k)$.* We work again at the Quot scheme level. Using the notations of Section 6, let $\mathcal{Q}_i$ be the corresponding Quot schemes, and R_i^s the open set of $\mathcal{Q}_i$ of stable points. Let f_i^s be the morphism from R_i^s to $\mathcal{M}_i$. In this situation, there exist universal bundles $\mathcal{U}_i^s$ on $R_i^s \times X$.

In this case we only have two strata, these are $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$ and $\mathscr{W}_{\mathcal{SE}_{\underline{n}}}$. We describe first $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$. As we have done earlier, we are able to construct a universal extension in the usual sense at the Quot scheme level. To this end, consider the projections $q^s : R_1^s \times R_2^s \times X \to R_1^s \times R_2^s$ and $p_i^s : R_1^s \times R_2^s \to R_i^s$ for $i = 1, 2$. Let $\mathcal{H}^s$ be the sheaf

$$\mathcal{R}^1 q_*^s(\mathscr{H}om((p_2^s \times id_X)^*\mathcal{U}_2^s, (p_1^s \times id_X)^*\mathcal{U}_1^s)).$$

Let $\mathbb{P}(\mathcal{H}^s)$ be the projectivization of $\mathcal{H}^s$. Let $\pi_P^s : \mathbb{P}(\mathcal{H}^s) \to \mathcal{U}_1^s \times \mathcal{U}_2^s$ and let $p_{\mathbb{P}(\mathcal{H}^s)} : \mathbb{P}(\mathcal{H}^s) \times X \to \mathbb{P}(\mathcal{H}^s)$ be the projection. We are again in the hypotheses of Remark 2.11, so there exists a vector bundle $\mathcal{F}^s$ over $\mathbb{P}(\mathcal{H}^s) \times X$ and an exact sequence

$$0 \to (\pi_P^s \times id_X)^*(p_1^s \times id_X)^*\mathcal{U}_1^s \otimes p_{\mathbb{P}(\mathcal{H}^s)}^*\mathcal{O}_P(1) \to \mathcal{F}^s \to (\pi_P^s \times id_X)^*(p_2^s \times id_X)^*\mathcal{U}_2^s \to 0, \tag{50}$$

which is universal in the sense of the projective version of Proposition 2.10.

Taking the dual of (50):

$$0 \to (\pi_P^s \times id_X)^*(p_2^s \times id_X)^*\mathcal{U}_2^{s\vee} \to \mathcal{F}^{s\vee} \to (\pi_P^s \times id_X)^*(p_1^s \times id_X)^*\mathcal{U}_1^{s\vee} \otimes p_{\mathbb{P}(\mathcal{H}^s)}^*\mathcal{O}_P(-1) \to 0,$$

and then $\mathcal{R}^i p_{\mathbb{P}(\mathcal{H}^s)*}$ we have

$$\begin{aligned} 0 \to \mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}(\pi_P^s \times id_X)^*(p_2^s \times id_X)^*\mathcal{U}_2^{s\vee} \to \mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}\mathcal{F}^{s\vee} \to \\ \to \mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}(\pi_P^s \times id_X)^*(p_1^s \times id_X)^*\mathcal{U}_1^{s\vee} \otimes \mathcal{O}_P(-1) \to 0. \end{aligned} \tag{51}$$

To simplify this extension, we introduce the following diagram

$$\begin{tikzcd} \mathbb{P}(\mathcal{H}^s) \times X \arrow[r, "\pi_P^s \times id_X"] \arrow[d, "p_{\mathbb{P}(\mathcal{H}^s)}"'] & \mathcal{R}_1^s \times \mathcal{R}_2^s \times X \arrow[r, "p_i^s \times id_X"] \arrow[d, "q^s"] & \mathcal{R}_i^s \times X \arrow[d, "\pi_i^s"] \\ \mathbb{P}(\mathcal{H}^s) \arrow[r, "\pi_P^s"] & \mathcal{R}_1^s \times \mathcal{R}_2^s \arrow[r, "p_i^s"] & \mathcal{R}_i^s \end{tikzcd}$$

Using again the base change formula we have

$$\mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}(\pi_P^s \times id_X)^*(p_i^s \times id_X)^*\mathcal{U}_i^{s\vee} \cong \pi_P^{s*} p_i^{s*} \mathcal{R}^1 \pi_{i*}^s \mathcal{U}_i^{s\vee}$$

so the extension (51) is

$$0 \to \pi_P^{s*} p_2^{s*} \mathcal{R}^1 \pi_{2*}^s \mathcal{U}_2^{s\vee} \to \mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}\mathcal{F}^{s\vee} \to \pi_P^* p_1^{s*} \mathcal{R}^1 \pi_{1*}^s \mathcal{U}_1^{s\vee} \otimes \mathcal{O}_P(-1) \to 0. \tag{52}$$

Consider now the set $W_{\mathcal{E}_{\underline{n}}} := \{(e_1, e_2, e) \text{ where } (e_1, e_2) \in \mathcal{R}_1^s \times \mathcal{R}_2^s \text{ and } e \in \mathbb{P}(\mathcal{H}^s_{(e_1,e_2)})\}$. Consider the Grassmann bundle of k-planes of the bundle $\mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}\mathcal{F}^{s\vee}$, let $\mathrm{Gr}(k, \mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}\mathcal{F}^{s\vee})$. For every point $w \in W_{\mathcal{E}_{\underline{n}}}$ we define the following determinantal variety

$$V_w := \left\{\pi \in \mathrm{Gr}(k, \mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}\mathcal{F}^{s\vee})_w : \dim\left(\pi \cap (\pi_P^{s*} p_2^{s*} \mathcal{R}^1 \pi_{2*}^s \mathcal{U}_2^{s\vee})_w\right) \geqslant k(1 - \frac{n_1}{n-k})\right\}.$$

Let $V_{\mathcal{E}_{\underline{n}}} := \coprod_{w \in W_{\mathcal{E}_{\underline{n}}}} V_w \subseteq \coprod_{w \in W_{\mathcal{E}_{\underline{n}}}} \mathrm{Gr}(k, \mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}\mathcal{F}^{s\vee})_w$, this is a family of determinantal varieties.

Now, from the proofs of Proposition 6.5 and Proposition 6.14 (i), we have that $\mathbb{P}(\mathcal{H}^s)/PGL(N_1) \times PGL(N_2)$ is a projective fibration over $\mathcal{M}_1 \times \mathcal{M}_2$. Because the scheme $V_{\mathcal{E}_{\underline{n}}}$ is closed and invariant under the action of $PGL(N_1) \times PGL(N_2)$, using Kempf's descent Lemma, $V_{\mathcal{E}_{\underline{n}}}$ descends to a projective scheme over $\mathbb{P}(\mathcal{H}^s)/PGL(N_1) \times PGL(N_2)$, which we call $\mathcal{V}_{\mathcal{E}_{\underline{n}}}$. If we denote by $\mathcal{V}^c_{\mathcal{E}_{\underline{n}}}$ the complement of $\mathcal{V}_{\mathcal{E}_{\underline{n}}}$ in

$$\coprod_{w \in W_{\mathcal{E}_{\underline{n}}}} \mathrm{Gr}(k, \mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}\mathcal{F}^{s\vee})_w / PGL(N_1) \times PGL(N_2),$$

we have the following

Theorem 7.5. *The stratum $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$ is identified with $\mathcal{V}^c_{\mathcal{E}_{\underline{n}}}$.*

Proof. This follows from the previous construction and Theorem 4.6. □

Regarding the stratum $\mathscr{W}_{\mathcal{SE}_{\underline{n}}}$, one may consider the universal bundle $(p_1^s \times id_X)^*\mathcal{U}_1^s \oplus (p_2^s \times id_X)^*\mathcal{U}_2^s$ over $R_1^s \times R_2^s \times X$. Consider the Grassmann bundle of k-planes of the bundle $\mathcal{R}^1 q_*^s((p_1^s \times id_X)^*\mathcal{U}_1^{s\vee} \oplus (p_2^s \times id_X)^*\mathcal{U}_2^{s\vee})$, let $\mathrm{Gr}^{\mathcal{SE}_{\underline{n}}} := \mathrm{Gr}(k, \mathcal{R}^1 q_*^s((p_1^s \times id_X)^*\mathcal{U}_1^{s\vee} \oplus (p_2^s \times id_X)^*\mathcal{U}_2^{s\vee}))$. For every point $w = (r_1, r_2) \in R_1^s \times R_2^s$ we define the determinantal varieties

$$V_w^1 := \big\{\pi \in \mathrm{Gr}_w^{\mathcal{SE}_{\underline{n}}} : \dim\big(\pi \cap H^1(\mathcal{U}_2^{s\vee}|_{\{r_2\}\times X})\big) \geqslant k(1 - \frac{n_1}{n-k})\big\},$$

and

$$V_w^2 := \big\{\pi \in \mathrm{Gr}_w^{\mathcal{SE}_{\underline{n}}} : \dim\big(\pi \cap H^1(\mathcal{U}_1^{s\vee}|_{\{r_1\}\times X})\big) \geqslant k(1 - \frac{n-k-n_1}{n-k})\big\}.$$

Let $V_{\mathcal{SE}_{\underline{n}}} := \coprod_{w \in R_1^s \times R_2^s} (V_w^1 \cup V_w^2) \subseteq \coprod_{w \in R_1^s \times R_2^s} \mathrm{Gr}_w^{\mathcal{SE}_{\underline{n}}}$, this is again a family of determinantal varieties. Using a descent argument, $V_{\mathcal{SE}_{\underline{n}}}$ descends to a closed scheme over $\mathcal{M}_1 \times \mathcal{M}_2$, which we call $\mathcal{V}_{\mathcal{SE}_{\underline{n}}}$. If we denote by $\mathcal{V}^c_{\mathcal{SE}_{\underline{n}}}$ the complement of $\mathcal{V}_{\mathcal{SE}_{\underline{n}}}$ in $\coprod_{w \in R_1^s \times R_2^s} \mathrm{Gr}_w^{\mathcal{SE}_{\underline{n}}}/PGL(N_1) \times PGL(N_2)$,we have the following

Theorem 7.6. *The stratum $\mathscr{W}_{\mathcal{SE}_{\underline{n}}}$ is identified with $\mathcal{V}^c_{\mathcal{SE}_{\underline{n}}}$.*

7.2.2. *The case when $n_1 = \frac{1}{2}(n-k)$.* For $\mathcal{E}_{\underline{n}}$ and $\mathcal{SE}_{\underline{n}}$ the construction is the same as before. For $\mathcal{E}'_{\underline{n}}$ and $\mathcal{SE}'_{\underline{n}}$, we only do the construction in the case in which the invariants are coprime, the remaining cases follow easily from the forthcoming construction and the previous one.

We do first $\mathscr{W}_{\mathcal{E}'_{\underline{n}}}$. We need a "universal" extension, but in this case, as we saw in Paragraph 6.28, it does not exist. To solve this problem, in Theorem 6.29 we proved the existence of a family $(e_p)_{p\in P}$ of extensions of ${q'_P}^*(p_1 \times id_X)^*\mathcal{P}$ by ${q'_P}^*(p_2 \times id_X)^*\mathcal{P} \otimes {p'_P}^*\mathcal{O}_P(1)$ over $P = \mathbb{P}(\mathscr{E}xt^1_{q'}(\Delta'^*(p_2 \times id_X)^*\mathcal{P}, \Delta'^*(p_1 \times id_X)^*\mathcal{P})^*)$ which is universal in the sense of 2.8.

This means that we have a local universal family of extensions, instead of the universal extension that we were allowed to construct in the case of the stratum induced by $\mathcal{E}_{\underline{n}}$. We denote $\mathcal{M}_\infty = \mathcal{M}$. So as we did in the case of $\mathcal{E}_{\underline{n}}$, we consider the set $W := \{(m, h) : m \in \mathcal{M}$ and $h \in \mathbb{P}(\mathcal{R}^1 q'_* \Delta'^* \mathscr{H}om((p_2 \times id_X)^*\mathcal{P}, (p_1 \times id_X)^*\mathcal{P})_m)\}$. For every point $w \in W$ we have an extension

$$0 \to F' \otimes \mathcal{O}_P(1)_m \to F \to F' \to 0,$$

taking the dual and H^i we get

$$0 \to H^1(F'^\vee) \to H^1(F^\vee) \to H^1(F'^\vee) \otimes \mathcal{O}_P(-1)_m \to 0.$$

We define the following variety

$$V_w := \big\{\pi \in \mathrm{Gr}(k, H^1(F^\vee)) : dim\big(\pi \cap H^1(F'^\vee)\big) \geqslant \frac{k}{2}\big\},$$

and let $\mathcal{V}_{\mathcal{E}'_{\underline{n}}} := \coprod_{w \in W} V_w \subseteq \coprod_{w\in W} \mathrm{Gr}(k, H^1(F^\vee))$.

Theorem 7.7. *The stratum $\mathscr{W}_{\mathcal{E}'_{\underline{n}}}$ is identified with $\mathcal{V}^c_{\mathcal{E}'_{\underline{n}}}$.*

Regarding the stratum $\mathscr{W}_{\mathcal{SE}'_{\underline{n}}}$, we have the universal bundle $(p_1 \times id_X)^*\mathcal{P} \oplus (p_2 \times id_X)^*\mathcal{P}$ over $\mathcal{M} \times X$. For every $m \in \mathcal{M}$, each $f \in \mathbb{P}^1$ defines a morphism of vector bundles

$$Q \xrightarrow{f} Q \oplus Q,$$

where $Q = \mathcal{P}|_{\{m\}\times X}$. Taking the dual and H^i we get $H^1(f^\vee) : H^1(Q^\vee) \oplus H^1(Q^\vee) \to H^1(Q^\vee)$. We then define the following variety

$$V_m^f := \{\pi \in \mathrm{Gr}(k, H^1(Q^\vee) \oplus H^1(Q^\vee)) : \dim\left(\pi \cap \ker H^1(f^\vee)\right) \geqslant \frac{k}{2}\},$$

and let $\mathcal{V}_{\mathcal{S}\mathcal{E}'_{\underline{n}}} := \coprod_{(f,m)\in\mathbb{P}^1\times\mathcal{M}} V_m^f \subseteq \coprod_{(f,m)\in\mathbb{P}^1\times\mathcal{M}} \mathrm{Gr}(k, H^1(Q^\vee) \oplus H^1(Q^\vee))$.

Theorem 7.8. *The stratum $\mathscr{W}_{\mathcal{S}\mathcal{E}'_{\underline{n}}}$ is identified with $\mathcal{V}^c_{\mathcal{S}\mathcal{E}'_{\underline{n}}}$.*

Remark 7.9. Using the construction above we see that one may describe the varieties corresponding to our strata at the Quot scheme level as locally trivial fiber bundles (in the Zariski topology). For instance, if we look at the stratum $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$, this is isomorphic to the descended variety corresponding to $W_{\mathcal{E}_{\underline{n}}}$ by the morphism

$$f_1^s \times f_2^s : \mathcal{R}_1^s \times \mathcal{R}_1^s \to \mathcal{M}_1 \times \mathcal{M}_2.$$

The variety $W_{\mathcal{E}_{\underline{n}}}$ is isomorphic to a locally trivial fibration (in the Zariski topology) over the projective fibration $\mathbb{P}(\mathcal{H}^s)$, over $\mathcal{R}_1^s \times \mathcal{R}_2^s$ that appears in the proof of Proposition 6.5. The fiber of our locally trivial fiber bundle is $V^c = V_w^c$, that is the complement of $V = V_w$ in $\mathrm{Gr}(k, \mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}\mathcal{F}^{s\vee})_w \cong \mathrm{Gr}(k, d + (n - k)(g-1))$. Moreover, both fibrations are invariant for the action of $PGL(N_1) \times PGL(N_2)$. For the rest of the strata one gets the same sort of description.

7.3. **Irreducibility of the strata.** In this subsection we will prove that the strata we have defined earlier in this paper are irreducible.

Theorem 7.10. *The strata described in Definition 7.1 are irreducible.*

Proof. The irreducibility condition for $\mathscr{W}^1$ comes directly from Proposition 4.5. The argument we use to prove that the rest of our strata are irreducible is the same for every stratum so we prove it for the simplest case. In Proposition 6.14 (i) we proved that when $n_1 \neq \frac{1}{2}(n-k)$ the space $\mathcal{E}_{\underline{n}}$ is isomorphic to a projective bundle over $\mathcal{M}_1 \times \mathcal{M}_2$ of constant dimension. Now, because $\mathcal{M}_1 \times \mathcal{M}_2$ is irreducible, we have that $\mathcal{E}_{\underline{n}}$ is also irreducible. By Theorem 7.5 we have defined an open family of extensions within $\mathcal{E}_{\underline{n}}$, hence this family is again irreducible and maps into $G_L(n,d,k)$. Its image is irreducible and is identified with $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$. □

8. Hodge–Poincaré polynomials

We use Deligne's extension of Hodge theory which applies to varieties which are not necessarily compact, projective or smooth (see [D1], [D2] and [D3]). We start by giving a review of the notions of pure Hodge structure, mixed Hodge structure, Hodge–Deligne and Hodge–Poincaré polynomials under these general hypotheses.

Definition 8.1. A *pure Hodge structure of weight m* is given by a finite dimensional $\mathbb{Q}$–vector space $H_{\mathbb{Q}}$ and a finite decreasing filtration F^p of $H = H_{\mathbb{Q}} \otimes \mathbb{C}$

$$H \supset \ldots \supset F^p \supset \ldots \supset (0),$$

called *the Hodge filtration*, such that $H = F^p \oplus \overline{F^{m-p+1}}$ for all p. When $p+q = m$, if we set $H^{p,q} = F^p \cap \overline{F^q}$, the condition $H = F^p \oplus \overline{F^{m-p+1}}$ for all p implies an equivalent definition for a pure Hodge structure. That is, a decomposition

$$H = \bigoplus_{p+q=m} H^{p,q}$$

satisfying that $H^{p,q} = \overline{H^{q,p}}$, where $\overline{H^{q,p}}$ is the complex conjugate of $H^{q,p}$. The relation between the two equivalent definitions is the following: Given a filtration $\{F^p\}_p$ we obtain a decomposition by considering $H^{p,q} = F^p \cap \overline{F^q}$. Given a decomposition $\{H^{p,q}\}_{p,q}$, this defines a filtration as above by $F^p = \bigoplus_{i\geq p} H^{i,m-i}$.

The n-cohomology group of a smooth projective variety $H^n(X)$ carries a pure Hodge structure of weight n. If $\Omega^\bullet_X$ denote the complex of holomorphic differential forms, and $(\Omega^\bullet_X)^{\geq p}$ is the subcomplex of forms of degree greater than or equal to p. One has that $H^n(X,\mathbb{C}) = \mathbb{H}(X,\Omega^\bullet_X)$. The role of the Hodge filtration is played here by the following filtration.

$$F^p = \mathrm{Im}(\mathbb{H}^n(X,(\Omega^\bullet_X)^{\geq p}) \to \mathbb{H}^n(X,\Omega^\bullet_X)).$$

8.2. A morphism of Hodge structures is a map $f_{\mathbb{Q}} : H_{\mathbb{Q}} \to H'_{\mathbb{Q}}$ such that $f_{\mathbb{C}}(F^pH) \subset F^pH'$ for all p, where $f_{\mathbb{C}} = f_{\mathbb{Q}} \otimes \mathbb{C}$ and F^pH is the *pth*–element in the Hodge filtration of H. When the Hodge structures have the same weight, $f_{\mathbb{Q}}$ strictly preserves the filtration, that is

$$\mathrm{Im}(f_{\mathbb{C}}) \cap F^pH' = f_{\mathbb{C}}(F^pH).$$

It is also known that for a given weight, the pure Hodge structures form an abelian category.

Definition 8.3. A *mixed Hodge structure* consists of a finite dimensional $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$, an increasing filtration W_l of $H_{\mathbb{Q}}$, called *the weight filtration*

$$\ldots \subset W_l \subset \ldots \subset H_{\mathbb{Q}},$$

and the Hodge filtration F^p of $H = H_{\mathbb{Q}} \otimes \mathbb{C}$, where the filtrations $F^pGr^W_l$ induced by F^p on

$$Gr^W_l = (W_lH_{\mathbb{Q}}/W_{l-1}H_{\mathbb{Q}}) \otimes \mathbb{C} = W_lH/W_{l-1}H$$

give a pure Hodge structure of weight l. Here $F^pGr^W_l$ is given by

$$(W_lH \cap F^p + W_{l-1}H)/W_{l-1}H.$$

8.4. A morphism of type (r,r) between mixed Hodge structures, $H_{\mathbb{Q}}$ with filtrations W_m and F^p, and $H'_{\mathbb{Q}}$ with W'_l and F'^q, is given by a linear map

$$L : H_{\mathbb{Q}} \to H'_{\mathbb{Q}}$$

satisfying $L(W_m) \subset W'_{m+2r}$ and $L(F^p) \subset F'^{p+r}$. Any such morphism is then strict in the sense that $L(F^p) = F'^{p+r} \cap \mathrm{Im}(L)$, and the same for the weight filtration.

Definition 8.5. A morphism of type $(0,0)$ between mixed Hodge structures, is called *a morphism of mixed Hodge structures.*

Our main interests in this paper are the cohomology groups $H^k(X,\mathbb{Q})$ of a complex variety X which may be singular and not projective. Deligne proved that these groups carry a mixed Hodge structure (see [D1], [D2] and [D3]). Associated to the Hodge filtration and the weight filtration we can consider the quotients $Gr^W_l = W_l/W_{l-1}$ of Definition 8.3, and for the Hodge filtration $Gr^p_FGr^W_l = F^pGr^W_l/F^{p+1}Gr^W_l$. Deligne also proved that the cohomology groups with compact support, we denote them by $H^k_c(X)$, carry a mixed Hodge structure (see [D1], [D2] and [D3]). We can then define the Hodge–Deligne numbers of X as follows

Definition 8.6. For a complex algebraic variety X, not necessarily smooth, compact or irreducible, we define its *Hodge–Deligne numbers* as

$$h^{p,q}(H^k_c(X)) = \dim Gr^p_FGr^W_{p+q}H^k_c(X).$$

We may introduce the following Euler characteristic

$$\chi^c_{p,q}(X) = \sum_k (-1)^k h^{p,q}(H^k_c(X)). \tag{53}$$

We write $\chi_{p,q}(X)$ for the Euler characteristic (53) of $H^k(X)$. Then under the hypothesis of X being smooth of dimension n, Poincaré duality tells us that

$$\chi^c_{p,q}(X) = \chi_{n-p,n-q}(X).$$

We are now ready to define the Hodge–Deligne polynomial.

Definition 8.7 ([DK])**.** For any complex algebraic variety X, we define its *Hodge–Deligne polynomial* (or virtual Hodge polynomial) as

$$\mathcal{H}(X)(u,v) = \sum_{p,q} (-1)^{p+q} \chi^c_{p,q}(X) u^p v^q \in \mathbb{Z}[u,v].$$

Danilov and Khovanskiǐ ([DK]) observed that $\mathcal{H}(X)(u,v)$ coincides with the classical Hodge polynomial when X is smooth and projective. Note that under these hypotheses, the mixed Hodge structure on $H^k_c(X)$ is pure of weight k, so

$$Gr^W_m H^k_c(X) = \begin{cases} H^k(X) \ \text{ if } \ m = k. \\ 0 \ \text{ if } \ m \neq k. \end{cases}$$

Then

$$\mathcal{H}(X)(u,v) = \sum_{p,q} h^{p,q}(X) u^p v^q, \tag{54}$$

where $h^{p,q}(X) = h^{p,q}(H^{p+q}(X))$ are the classical Hodge numbers of X and (54) the classical Hodge polynomial.

We may define another polynomial using the Euler characteristic $\chi_{p,q}(X)$ for rational cohomology groups without compact support. As we have already said Deligne proved that these groups carry a mixed Hodge structure with the usual given associated filtrations.

Definition 8.8. For a complex algebraic variety X, not necessarily smooth, compact or irreducible, we define its *Hodge–Poincaré numbers* as

$$h^{p,q}(H^k(X)) = \dim Gr^p_F Gr^W_{p+q} H^k(X).$$

We are ready now to define the Hodge–Poincaré polynomial.

Definition 8.9. For any complex algebraic variety X, we define its *Hodge–Poincaré polynomial* as

$$HP(X)(u,v) = \sum_{p,q} (-1)^{p+q} \chi_{p,q}(X) u^p v^q = \sum_{p,q,k} (-1)^{p+q+k} h^{p,q}(H^k(X)) u^p v^q.$$

Remark 8.10. When our algebraic variety X is smooth, Poincaré duality gives us the following functional identity relating Hodge–Deligne and Hodge–Poincaré polynomials

$$\mathcal{H}(X)(u,v) = (uv)^{\dim_{\mathbb{C}} X} \cdot HP(X)(u^{-1}, v^{-1}) \tag{55}$$

where $\dim_{\mathbb{C}} X$ denotes the complex dimension of X.

Let $b^k(X) = \dim H^k(X)$ be the k–Betti number of the variety X and let $P_X(t) = \sum_k b^k(X) t^k$ be its Poincaré polynomial. If X is not only smooth, but also projective, the Betti numbers of X satisfy

$$b^k(X) = \sum_{p+q=k} h^{p,q}(H^k(X)) \tag{56}$$

so that

$$P_X(t) = \sum_k b^k(X)t^k = \mathcal{H}(X)(t,t) = HP(X)(t,t). \tag{57}$$

Hodge–Deligne polynomials are very useful because of their very nice properties. Now we introduce some results that will be very helpful to do our computations. In [Du] Durfee proved that if $X = \cup_i X_i$ and $Y = \cup_i Y_i$ are smooth projective varieties that are disjoint unions of locally closed subvarieties, such that $X_i \cong Y_i$ for all i, then X and Y have the same Betti numbers. Using the properties of Hodge–Deligne polynomials, in particular their relation with virtual Poincaré polynomials, one may prove that this is also true for the Hodge numbers of X and Y. Here we are using the following extension of Durfee's result

Theorem 8.11 ([MOV1], Theorem 2.2)**.** *Let X be a complex variety. Suppose that X is a finite disjoint union $X = \cup_i X_i$, where X_i are locally closed subvarieties. Then*

$$\mathcal{H}(X)(u,v) = \sum_i \mathcal{H}(X_i)(u,v).$$

Another result from [MOV1] that will be useful for our computations when we are dealing with fibrations is

Lemma 8.12 ([MOV1], Lemma 2.3)**.** *Suppose that $\pi : X \to Y$ is an algebraic fiber bundle with fiber F which is locally trivial in the Zariski topology, then*

$$\mathcal{H}(X)(u,v) = \mathcal{H}(F)(u,v) \cdot \mathcal{H}(Y)(u,v).$$

In this paper we consider varieties acted on by algebraic groups. Then, we need a cohomology theory that captures all the information given by the action of the group. Namely equivariant cohomology. Hodge–Poincaré polynomials can be extended to analogous polynomials for equivariant cohomology groups. We shall call this new series the equivariant Hodge–Poincaré series.

If X is an algebraic variety acted on by a group G, consider $EG \to BG$ a universal classifying bundle for G, where $BG = EG/G$ is the *classifying space* of G and EG is the *total space* of G. We form the space $X \times_G EG$ which is defined to be the quotient space of $X \times EG$ by the equivalence relation $(x, e \cdot g) \sim (g \cdot x, e)$. Then, the *equivariant cohomology ring* of X is the following

$$H^*_G(X) = H^*(X \times_G EG).$$

Although EG and BG are not finite-dimensional manifolds, there are natural Hodge structures on their cohomology. This is trivial in the case of EG. Deligne proved that there is a pure Hodge structure on $H^*(BG)$ and that $H^{p,q}(H^*(BG)) = 0$ for $p \neq q$ (see [D3] §9). We may regard EG and BG as increasing unions of finite-dimensional varieties $(EG)_m$ and $(BG)_m$ for $m \geq 1$ such that G acts freely on $(EG)_m$ with $(EG)_m/G = (BG)_m$ and the inclusions of $(EG)_m$ and $(BG)_m$ in EG and BG respectively induce isomorphisms of cohomology in degree less than m which preserve the Hodge structures. In the same way $X \times_G EG$ is the union of finite-dimensional varieties whose natural mixed Hodge structures induce a natural mixed Hodge structure on $H^n(X \times_G EG)$. Using that we have the following

Definition 8.13. We define the *equivariant Hodge–Poincaré numbers* of X as

$$h^{p,q;n}_G(X) = h^{p,q}(H^n(X \times_G EG)).$$

We are ready now to define the equivariant Hodge–Poincaré series.

Definition 8.14. For any complex algebraic variety X acted on by an algebraic group G, we define its *equivariant Hodge–Poincaré series* as

$$HP_G(X)(u,v) = \sum_{p,q,k} (-1)^{p+q+k} h_G^{p,q;k}(X) u^p v^q.$$

8.15. Suppose now that G is connected. The relationship between cohomology and equivariant cohomology is accounted for by a Leray spectral sequence for the fibration

$$X \times_G EG \to BG \tag{58}$$

whose fiber is X. The E_2-term of this spectral sequence is given by $E_2^{p,q} = H^p(X) \otimes H^q(BG)$ which abuts to $H_G^{p+q}(X)$. This spectral sequence preserves Hodge structures.

If X is a nonsingular projective variety that is acted on linearly by a connected complex reductive group G, one has that the fibration (58) is cohomologically trivial over $\mathbb{Q}$ (see [K] Theorem 5.8). Then

$$H_G^*(X) \cong H^*(X) \otimes H^*(BG). \tag{59}$$

This isomorphism is actually an isomorphism of mixed Hodge structures ([D3] 8.2.10). In order for fibration (58) to be cohomologically trivial it is not necessary to have so strong a condition for our variety X. For instance, if the variety X has cohomology groups of (p,p) type only, one can easily check that the Leray spectral sequence in this case implies that $\oplus_{p+q=m} E_2^{p,q} = H_G^m(X)$. This is the case of varieties such as complex affine spaces or those that admit a decomposition as a union of cells. An example of the latter is the Grassmannian.

We have another fibration, that is

$$X \times_G EG \to X/G$$

with fiber EG. When G acts freely on X, that is the stabilizer of every point is trivial, then it induces the isomorphism

$$H^*(X \times_G EG) \cong H^*(X/G). \tag{60}$$

Hence, if X is finite-dimensional and G acts freely on it, $HP_G(X)(u,v)$ is a polynomial.

We need the following result from [GM] for future computations.

Lemma 8.16. *Let $Y \to Z$ be a locally trivial fibration in the Zariski topology with fibre F, and such that it is compatible with respect to the action of the group G that acts on Y and Z respectively. Assume that Y and Z are smooth varieties. Then*

$$HP_G(Y)(u,v) = HP_G(Z)(u,v) \cdot HP(F)(u,v).$$

We are ready now to compute the Hodge–Deligne polynomials of our strata. In the rest of the section we will describe how we can do it for the case in which we have two components in the type we use to define the stratification. When $\underline{n} = (n_1, n-k-n_1)$, we proved that the stratum can be described as a complement of a determinantal variety. Our strategy could be understood by looking at what happens at the stratum $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$ when $n_1 \neq \frac{1}{2}(n-k)$. The remaining cases are analogous.

Theorem 8.17. *Using the notations of Subsection 7.2, the stratum $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$ for the type $\underline{n}=(n_1, n-k-n_1)$ has the following Hodge–Poincaré polynomial*

$$\begin{aligned}
HP(\mathscr{W}_{\mathcal{E}_{\underline{n}}})(u,v) &= HP(\mathcal{M}(n_1,d_1))(u,v)\cdot HP(\mathcal{M}(n-k-n_1,d-d_1))(u,v)\\
&\cdot \frac{1-(uv)^{n_1\cdot(n-k-n_1)\cdot(g-1)}}{1-uv}\cdot\Bigg[\frac{(1-(uv)^{N-k+1})\cdot\ldots\cdot(1-(uv)^{N})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{k})}-\\
&-\sum_{\mu=\lceil k(1-\frac{n_1}{n-k})\rceil}^{\min\{k,j\}}(uv)^{\mu(N-k-j+\mu)}\cdot\frac{(1-(uv)^{N-j-k+\mu+1})\cdot\ldots\cdot(1-(uv)^{N-j})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{k-\mu})}\cdot\\
&\cdot\frac{(1-(uv)^{j-\mu+1})\cdot\ldots\cdot(1-(uv)^{j})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{\mu})}\Bigg],
\end{aligned}$$

where $N=d+(n-k)(g-1)$ and $j=d-d_1+(n-k-n_1)(g-1)$. The numbers d_1 and $d-d_1$ must satisfy the following identity:

$$\frac{d_1}{n_1}=\frac{d-d_1}{n-k-n_1}.$$

Proof. In Remark 7.9 we saw that $W_{\mathcal{E}_{\underline{n}}}$ may be described as a locally trivial fiber bundle (in the Zariski topology) over $\mathbb{P}(\mathcal{H}^s)$, where $\mathbb{P}(\mathcal{H}^s)$ is the projective fibration over $\mathcal{R}_1^s\times\mathcal{R}_2^s$ that appears in the proof of Proposition 6.5, with fiber the complement of V, we denote it by V^c, in $\mathrm{Gr}(k,d+(n-k)(g-1))$. We also saw that both fibrations are $PGL(N_1)\times PGL(N_2)$-invariant, for certain N_1 and N_2. Note that $\mathbb{P}(\mathcal{H}^s)$ is actually a projective fibration with fiber the projective space of dimension $(n_1)(n-k-n_1)(g-1)-1$. We label $N=h^1(F^\vee)=d+(n-k)(g-1)$. Using then Lemma 8.16 we have that

$$\begin{aligned}
&HP_{PGL(N_1)\times PGL(N_2)}(W_{\mathcal{E}_{\underline{n}}})(u,v)=HP_{PGL(N_1)\times PGL(N_2)}(\mathbb{P}(\mathcal{H}^s))(u,v)\cdot HP(V^c)(u,v)= \qquad (61)\\
&=HP_{PGL(N_1)\times PGL(N_2)}(\mathcal{R}_1^s\times\mathcal{R}_2^s)(u,v)HP(\mathbb{P}^{n_1\cdot(n-k-n_1)\cdot(g-1)-1})(u,v)\cdot HP(V^c)(u,v).
\end{aligned}$$

Now, the varieties $W_{\mathcal{E}_{\underline{n}}}$ and $\mathcal{R}_1^s\times\mathcal{R}_2^s$ are closed under the action of $PGL(N_1)\times PGL(N_2)$. This group is connected and the action is actually free then the stabilizers are trivial. Then we may apply paragraph 8.15. We obtain that identities (59) and (60) hold, then

$$H^*_{PGL(N_1)\times PGL(N_2)}(W_{\mathcal{E}_{\underline{n}}})\cong H^*(W_{\mathcal{E}_{\underline{n}}}/PGL(N_1)\times PGL(N_2))\cong H^*(\mathscr{W}_{\mathcal{E}_{\underline{n}}})$$

and

$$\begin{aligned}
H^*_{PGL(N_1)\times PGL(N_2)}(\mathcal{R}_1^s\times\mathcal{R}_2^s)&\cong H^*(\mathcal{R}_1^s\times\mathcal{R}_2^s/PGL(N_1)\times PGL(N_2))\cong\\
&\cong H^*(\mathcal{M}(n_1,d_1)\times\mathcal{M}(n-k-n_1,d-d_1))\cong\\
&\cong H^*(\mathcal{M}(n_1,d_1))\otimes H^*(\mathcal{M}(n-k-n_1,d-d_1)),
\end{aligned}$$

using Künneth formula. These are isomorphisms of mixed Hodge structures, so induce the following identity of Hodge–Poincaré polynomials

$$HP_{PGL(N_1)\times PGL(N_2)}(W_{\mathcal{E}_{\underline{n}}})(u,v)=HP(\mathscr{W}_{\mathcal{E}_{\underline{n}}})(u,v)$$

and

$$HP_{PGL(N_1)\times PGL(N_2)}(\mathcal{R}_1^s\times\mathcal{R}_2^s)(u,v)=HP(\mathcal{M}(n_1,d_1))(u,v)\cdot HP(\mathcal{M}(n-k-n_1,d-d_1))(u,v).$$

Now, the Hodge–Poincaré polynomial of the projective space is $HP(\mathbb{P}^n)(u,v) = \frac{1-(uv)^{n+1}}{1-uv}$ for every n. Substituting these in (61) one obtains the following identity of Hodge–Poincaré polynomials

$$HP(\mathscr{W}_{\mathcal{E}_{\underline{n}}})(u,v) =$$
$$= HP(\mathcal{M}(n_1,d_1))(u,v) \cdot HP(\mathcal{M}(n-k-n_1,d-d_1))(u,v) \cdot \frac{1-(uv)^{n_1\cdot(n-k-n_1)\cdot(g-1)}}{1-uv} \cdot HP(V^c)(u,v).$$

Regarding $HP(V^c)(u,v)$, the variety $V = V_w$ where w is a point in $W_{\mathcal{E}_{\underline{n}}} := \{(e_1,e_2,e)$ where $(e_1,e_2) \in \mathcal{R}_1^s \times \mathcal{R}_2^s$ and $e \in \mathbb{P}(\mathcal{H}^s_{(e_1,e_2)})\}$ using the notations of Subsection 7.2. The variety V is actually independent of the point w and is equal to

$$V = V_w := \big\{\pi \in \mathrm{Gr}(k, \mathcal{R}^1 p_{\mathbb{P}(\mathcal{H}^s)*}\mathcal{F}^{s\vee})_w : \dim\big(\pi \cap (\pi_P^{s*} p_2^{s*}\mathcal{R}^1\pi^s_{2*}\mathcal{U}_2^{s\vee})_w\big) \geqslant k(1-\frac{n_1}{n-k})\big\},$$

let $(\pi_P^{s*} p_2^{s*}\mathcal{R}^1\pi^s_{2*}\mathcal{U}_2^{s\vee})_w = H^1(Q_2^\vee)$ and $j = h^1(Q_2^\vee) = d - d_1 + (n-k-n_1)(g-1)$. Analogously we denote $\pi_P^* p_1^{s*}\mathcal{R}^1\pi^s_{1*}\mathcal{U}_1^{s\vee} \otimes \mathcal{O}_P(-1) = H^1(F_1^\vee)$ and $h^1(F_1^\vee) = n_1(g-1)+d_1$. Then V can be written as

$$V := \coprod_{\mu=\lceil k(1-\frac{n_1}{n-k})\rceil}^{\min\{k,j\}} \big\{\pi \in \mathrm{Gr}(k,N) : \dim\big(\pi \cap H^1(Q_2^\vee)\big) = \mu\big\}, \tag{62}$$

and denote $V^\mu := \big\{\pi \in \mathrm{Gr}(k,N) : \dim\big(\pi \cap H^1(Q_2^\vee)\big) = \mu\big\}$ for integers μ between $\lceil k(1-\frac{n_1}{n-k})\rceil$ and $\min\{k,j\}$.

For every μ, the variety V^μ is isomorphic to a fibration over $\mathrm{Gr}(k-\mu, N-j) \times \mathrm{Gr}(\mu, j)$ with fibre $\mathbb{C}^{(j-\mu)(k-\mu)}$. Then, we have the following identity of Hodge–Deligne polynomials

$$\mathcal{H}(V^\mu)(u,v) = \mathcal{H}(\mathrm{Gr}(k-\mu,N-j))(u,v) \cdot \mathcal{H}(\mathrm{Gr}(\mu,j))(u,v) \cdot \mathcal{H}(\mathbb{C}^{(j-\mu)(k-\mu)})(u,v). \tag{63}$$

Now, from Remark 8.10 one has that $HP(V^c)(u,v) = (uv)^{\dim_{\mathbb{C}} V^c}\mathcal{H}(V^c)(u^{-1},v^{-1})$. Using now Theorem 8.11 and applying again the previous identity relating Hodge–Poincaré and Hodge–Deligne polynomials, we obtain

$$\begin{aligned}
HP(V^c)(u,v) &= (uv)^{\dim_{\mathbb{C}} V^c}\mathcal{H}(V^c)(u^{-1},v^{-1}) = \\
&= (uv)^{\dim_{\mathbb{C}} V^c}\Big[\mathcal{H}(\mathrm{Gr}(k,N))(u^{-1},v^{-1}) - \sum_{\mu=\lceil k(1-\frac{n_1}{n-k})\rceil}^{\min\{k,j\}} \mathcal{H}(V^\mu)(u^{-1},v^{-1})\Big] = \\
&= (uv)^{\dim_{\mathbb{C}} V^c}\Big[(uv)^{-\dim_{\mathbb{C}}\mathrm{Gr}(k,N)} HP(\mathrm{Gr}(k,N))(u,v) - \\
&- \sum_{\mu=\lceil k(1-\frac{n_1}{n-k})\rceil}^{\min\{k,j\}} (uv)^{-k(N-k)+\mu(N-k-j+\mu)}\Big[HP(\mathrm{Gr}(k-\mu,N-j))(u,v)\cdot \\
&\cdot HP(\mathrm{Gr}(\mu,j))(u,v) \cdot HP(\mathbb{C}^{(j-\mu)(k-\mu)})(u,v)\Big]\Big].
\end{aligned} \tag{64}$$

The Grassmannian $\mathrm{Gr}(k,N)$ is a smooth projective variety. Note that the Hodge–Poincaré polynomial of the Grassmannian, $HP(\mathrm{Gr}(k,N))(u,v)$, is rather simple. The cohomology of the Grassmannian is integral, hence only types (p,p) occur. This fact implies that the identity (56) is in this case the following

$$b^{2p}(\mathrm{Gr}(k,N)) = h^{p,p}(H^{2p}(\mathrm{Gr}(k,N)))$$

so

$$HP(\mathrm{Gr}(k,N))(u,v)=\sum_p h^{p,p}(H^{2p}(\mathrm{Gr}(k,N)))u^pv^p=\frac{(1-(uv)^{N-k+1})\cdot\ldots\cdot(1-(uv)^N)}{(1-uv)\cdot\ldots\cdot(1-(uv)^k)}. \tag{65}$$

Moreover, the variety V^c is open in $\mathrm{Gr}(k,N)$ then they have the same dimension. In addition, it is not difficult to see that $HP(\mathbb{C}^m)(u,v)=1$ for all m. Substituting these in (64), we get

$$\begin{aligned}HP(V^c)(u,v)=&\frac{(1-(uv)^{N-k+1})\cdot\ldots\cdot(1-(uv)^N)}{(1-uv)\cdot\ldots\cdot(1-(uv)^k)}-\\&-\sum_{\mu=\lceil k(1-\frac{n_1}{n-k})\rceil}^{\min\{k,j\}}(uv)^{\mu(N-k-j+\mu)}\cdot\frac{(1-(uv)^{N-j-k+\mu+1})\cdot\ldots\cdot(1-(uv)^{N-j})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{k-\mu})}\cdot\\&\cdot\frac{(1-(uv)^{j-\mu+1})\cdot\ldots\cdot(1-(uv)^j)}{(1-uv)\cdot\ldots\cdot(1-(uv)^\mu)}.\end{aligned}$$

Then we conclude. □

Remark 8.18. Regarding the Hodge–Poincaré polynomial of the moduli space of stable bundles of rank n and degree d not everything is known. For $(n,d)=1$, the expression for $HP(\mathcal{M}(2,d))(u,v)$ can be deduced from Peter Newstead's article [N1], although it did not appear written out in this paper. The first time that this appeared in the literature is in the article [Ba] by S. del Baño Rollín. In [EK] R. Earl and F. Kirwan give an inductive formula for the Hodge–Poincaré polynomials of this moduli spaces, and in particular they compute it explicitly for some cases with rank different to 2. When $(n,d)\neq 1$ the Hodge–Deligne polynomial $\mathcal{H}(\mathcal{M}(2,d))(u,v)$ where $\mathcal{M}(2,d)$ is the moduli space of stable vector bundles of rank 2 and even degree, has been recently computed by Muñoz *et al.* (see [MOV2] Theorem 5.2) using its relation with certain moduli spaces of triples and by myself in [GM].

8.1. **Explicit computations for $n-k=2$.** Under this hypothesis we see that our coherent systems (E,V) of type (n,d,k) are coming from BGN extensions whose quotient bundle F has rank 2. Then the subbundles Q_1 and Q_2 are actually line bundles, hence the type in this case is $\underline{n}=(n_1,n-k-n_1)=(1,1)$. Bearing in mind the equality of the slopes, the degrees satisfy that $d_1=d/2=d-d_1$. Using the notations of Definition 7.1, we have the following decomposition

$$G_L(n,d,k)=\mathscr{W}^1\sqcup\mathscr{W}_{\mathcal{E}_{\underline{n}}}\sqcup\mathscr{W}_{\mathcal{E}'_{\underline{n}}}\sqcup\mathscr{W}_{\mathcal{SE}_{\underline{n}}}\sqcup\mathscr{W}_{\mathcal{SE}'_{\underline{n}}}. \tag{66}$$

Here $\mathscr{W}^1$ is the open stratum and classifies the coherent systems coming from a BGN extension of quotient being stable. The stratum $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$ classifies the cases in which the quotient bundle is the bundle in the middle of an extension of the following type

$$0\to L\to F\to L'\to 0 \tag{67}$$

that is a nonsplit extension and the line bundles L and L' are nonisomorphic. In the same fashion $\mathscr{W}_{\mathcal{E}'_{\underline{n}}}$ classifies the cases in which (67) satisfies that $L\cong L'$. The varieties $\mathscr{W}_{\mathcal{SE}_{\underline{n}}}$ and $\mathscr{W}_{\mathcal{SE}'_{\underline{n}}}$ are as before but for the bundle F being split.

We have two different cases, either $(n-k,d)=(2,d)=1$ or $(n-k,d)=(2,d)\neq 1$. The computations for these cases are done in the following theorems.

Theorem 8.19. *The Hodge–Deligne polynomial of the moduli space $G_L(n,d,k)$ for $(n-k,d)=(2,d)=1$ is*

$$\begin{aligned}\mathcal{H}(G_L(n,d,k))(u,v)=&(1+u)^g(1+v)^g\cdot\frac{(1+u^2v)^g(1+uv^2)^g-u^gv^g(1+u)^g(1+v)^g}{(1-uv)(1-u^2v^2)}\cdot\\&\cdot\frac{(1-(uv)^{2(g-1)+d-k+1})\cdot\ldots\cdot(1-(uv)^{2(g-1)+d})}{(1-uv)\cdot\ldots\cdot(1-(uv)^k)}\end{aligned}$$

Proof. Using Proposition 4.5 we have that when $(n-k,d)=1$, then $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$, $\mathscr{W}_{\mathcal{E}'_{\underline{n}}}$, $\mathscr{W}_{\mathcal{SE}_{\underline{n}}}$, and $\mathscr{W}_{\mathcal{SE}'_{\underline{n}}}$ are all empty and $G_L(n,d,k)$ is actually a Grassmann fibration on $\mathcal{M}(n-k,d)$ with fiber the Grassmannian $\mathrm{Gr}(k,d+(n-k)(g-1))$. Using now Lemma 8.12 we get that

$$\mathcal{H}(G_L(n,d,k))(u,v)=\mathcal{H}(\mathrm{Gr}(k,d+2(g-1)))(u,v)\cdot\mathcal{H}(\mathcal{M}(2,d))(u,v).$$

We already know what $\mathcal{H}(\mathrm{Gr}(k,d+2(g-1)))(u,v)$ looks like, the computation appears in (65). Regarding $\mathcal{H}(\mathcal{M}(2,d))(u,v)$, for d odd, using Lemma 3 and Corollary 5 of [EK] we get that

$$\mathcal{H}(\mathcal{M}(2,d))(u,v)=(1+u)^g(1+v)^g\cdot\frac{(1+u^2v)^g(1+uv^2)^g-u^gv^g(1+u)^g(1+v)^g}{(1-uv)(1-u^2v^2)},$$

so we conclude. ☐

Theorem 8.20. *The Hodge–Deligne polynomial of the moduli space $G_L(n,d,k)$ for $(n-k,d)=(2,d)\neq 1$ is*

$$\begin{aligned}\mathcal{H}(G_L(n,d,k))(u,v)=&\frac{1}{2(1-uv)(1-u^2v^2)}[2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g-\\&-(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2)-(1-u^2)^g(1-v^2)^g(1-uv)^2]\cdot\\&\cdot\frac{(1-(uv)^{2(g-1)+d-k+1})\cdot\ldots\cdot(1-(uv)^{2(g-1)+d})}{(1-uv)\cdot\ldots\cdot(1-(uv)^k)}+\\&+\Bigg[((1+u)^{2g}(1+v)^{2g}-(1+u)^g(1+v)^g)\cdot\frac{1-(uv)^{g-1}}{1-uv}+(1+u)^g(1+v)^g\cdot\frac{1-(uv)^g}{1-uv}+\\&+\frac{1}{2}(1+u)^{2g}(1+v)^{2g}+\frac{1}{2}(1-u^2)^g(1-v^2)^g\Bigg]\cdot\Bigg[\frac{(1-(uv)^{2(g-1)+d-k+1})\cdot\ldots\cdot(1-(uv)^{2(g-1)+d})}{(1-uv)\cdot\ldots\cdot(1-(uv)^k)}-\\&-\sum_{\mu=\frac{k}{2}}^{\min\{k,(g-1)+\frac{d}{2}\}}(u\cdot v)^{\mu(d/2+(g-1)-k+\mu)}\cdot\frac{(1-(uv)^{d/2+(g-1)-k+\mu+1})\cdot\ldots\cdot(1-(uv)^{d/2+g-1})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{k-\mu})}\cdot\\&\cdot\frac{(1-(uv)^{(g-1)+d/2-\mu+1})\cdot\ldots\cdot(1-(uv)^{(g-1)+d/2})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{\mu})}\Bigg]-\end{aligned}$$

$$- \left[\frac{1}{2}(1+u)^{2g}(1+v)^{2g} + \frac{1}{2}(1-u^2)^g(1-v^2)^g - (1-uv)(1+u)^g(1+v)^g\right] \cdot$$

$$\cdot \Bigg[\sum_{\mu=\frac{k}{2}}^{\min\{k,(g-1)+\frac{d}{2}\}} (u\cdot v)^{\mu(d/2+(g-1)-k+\mu)} \cdot \frac{(1-(uv)^{d/2+(g-1)-k+\mu+1})\cdot \ldots \cdot (1-(uv)^{d/2+g-1})}{(1-uv)\cdot \ldots \cdot (1-(uv)^{k-\mu})} \cdot$$

$$\cdot \frac{(1-(uv)^{(g-1)+d/2-\mu+1})\cdot \ldots \cdot (1-(uv)^{(g-1)+d/2})}{(1-uv)\cdot \ldots \cdot (1-(uv)^{\mu})}\Bigg] +$$

$$\left[\frac{1}{2}(1+u)^{2g}(1+v)^{2g} + \frac{1}{2}(1-u^2)^g(1-v^2)^g - (1-(1+uv)^2+(1+uv))(1+u)^g(1+v)^g\right] \cdot$$

$$\cdot \left[\frac{(1-(uv)^{d/2+(g-1)-k/2+1})^2 \cdot \ldots \cdot (1-(uv)^{d/2+g-1})^2}{(1-uv)^2 \cdot \ldots \cdot (1-(uv)^{k/2})^2}\right]$$

when k is even, and

$$\mathcal{H}(G_L(n,d,k))(u,v) = \frac{1}{2(1-uv)(1-u^2v^2)}[2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g -$$

$$- (1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2] \cdot$$

$$\cdot \frac{(1-(uv)^{2(g-1)+d-k+1})\cdot \ldots \cdot (1-(uv)^{2(g-1)+d})}{(1-uv)\cdot \ldots \cdot (1-(uv)^{k})} +$$

$$+ \Bigg[((1+u)^{2g}(1+v)^{2g} - (1+u)^g(1+v)^g)\cdot \frac{1-(uv)^{g-1}}{1-uv} + (1+u)^g(1+v)^g \cdot \frac{1-(uv)^g}{1-uv} +$$

$$+ \frac{1}{2}(1+u)^{2g}(1+v)^{2g} + \frac{1}{2}(1-u^2)^g(1-v^2)^g\Bigg] \cdot \Bigg[\frac{(1-(uv)^{2(g-1)+d-k+1})\cdot \ldots \cdot (1-(uv)^{2(g-1)+d})}{(1-uv)\cdot \ldots \cdot (1-(uv)^{k})} -$$

$$- \sum_{\mu=\lceil\frac{k}{2}\rceil}^{\min\{k,(g-1)+\frac{d}{2}\}} (u\cdot v)^{\mu(d/2+(g-1)-k+\mu)} \cdot \frac{(1-(uv)^{d/2+(g-1)-k+\mu+1})\cdot \ldots \cdot (1-(uv)^{d/2+g-1})}{(1-uv)\cdot \ldots \cdot (1-(uv)^{k-\mu})} \cdot$$

$$\cdot \frac{(1-(uv)^{(g-1)+d/2-\mu+1})\cdot \ldots \cdot (1-(uv)^{(g-1)+d/2})}{(1-uv)\cdot \ldots \cdot (1-(uv)^{\mu})}\Bigg] -$$

$$- \left[\frac{1}{2}(1+u)^{2g}(1+v)^{2g} + \frac{1}{2}(1-u^2)^g(1-v^2)^g - (1-uv)(1+u)^g(1+v)^g\right] \cdot$$

$$\cdot \Bigg[\sum_{\mu=\lceil\frac{k}{2}\rceil}^{\min\{k,(g-1)+\frac{d}{2}\}} (u\cdot v)^{\mu(d/2+(g-1)-k+\mu)} \cdot \frac{(1-(uv)^{d/2+(g-1)-k+\mu+1})\cdot \ldots \cdot (1-(uv)^{d/2+g-1})}{(1-uv)\cdot \ldots \cdot (1-(uv)^{k-\mu})} \cdot$$

$$\cdot \frac{(1-(uv)^{(g-1)+d/2-\mu+1})\cdot \ldots \cdot (1-(uv)^{(g-1)+d/2})}{(1-uv)\cdot \ldots \cdot (1-(uv)^{\mu})}\Bigg]$$

when k is odd.

Proof. Applying Theorem 8.11 to (66) we obtain the following identity

$$\mathcal{H}(G_L(n,d,k))(u,v) =$$
$$= \mathcal{H}(\mathscr{W}^1)(u,v) + \mathcal{H}(\mathscr{W}_{\mathcal{E}_{\underline{n}}})(u,v) + \mathcal{H}(\mathscr{W}_{\mathcal{E}'_{\underline{n}}})(u,v) + \mathcal{H}(\mathscr{W}_{\mathcal{E}\mathcal{S}_{\underline{n}}})(u,v) + \mathcal{H}(\mathscr{W}_{\mathcal{E}\mathcal{S}'_{\underline{n}}})(u,v).$$

As we did in the proof of Theorem 8.19, Proposition 4.5 tells us that when $(n,d,k)=1$, $\mathscr{W}^1$ is a Grassmann fibration on $\mathcal{M}(n-k,d)$ with fiber the Grassmannian $\mathrm{Gr}(k, d+(n-k)(g-1))$. Here, $(n-k,d)=(2,d)\neq 1$

then the Grassmann fibration is constructed at the Quot-scheme level since there is no a Poincaré bundle over $\mathcal{M}(2,d)$ (see [BG2], Proposition 4.4). This fibration at the Quot-scheme level is locally trivial in the Zariski topology, then, if we denote R^s the corresponding set of stable points and W the set corresponding to $\mathscr{W}^1$, at the Quot-scheme level, from Lemma 8.16 we have that

$$HP_{PGL(N)}(W)(u,v) = HP(\mathrm{Gr}(k, d+2(g-1)))(u,v) \cdot HP_{PGL(N)}(R^s)(u,v).$$

The action of $PGL(N)$ on W and R^s is free, then

$$HP(\mathscr{W}^1)(u,v) = HP(\mathrm{Gr}(k, d+2(g-1)))(u,v) \cdot HP(\mathcal{M}(2,d))(u,v).$$

Moreover, by Theorem 7.2 (c) we have that $\mathscr{W}^1$ is smooth, applying Remark 8.10 we get that

$$\mathcal{H}(\mathscr{W}^1)(u,v) = \mathcal{H}(\mathrm{Gr}(k, d+2(g-1)))(u,v) \cdot \mathcal{H}(\mathcal{M}(2,d))(u,v),$$

where $\mathcal{H}(\mathcal{M}(2,d))(u,v)$ is the Hodge–Deligne polynomial of the moduli space $\mathcal{M}(2,d)$ of stable vector bundles of rank 2 and even degree. This can be found in [MOV2], Theorem 5.2. The polynomial is

$$\begin{aligned}\mathcal{H}(\mathcal{M}(2,d))(u,v) = & \frac{1}{2(1-uv)(1-u^2v^2)}[2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - \\ & -(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2)-(1-u^2)^g(1-v^2)^g(1-uv)^2]\end{aligned}$$

then we obtain

$$\begin{aligned}\mathcal{H}(\mathscr{W}^1)(u,v) = & \frac{1}{2(1-uv)(1-u^2v^2)}[2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - \\ & -(1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2)-(1-u^2)^g(1-v^2)^g(1-uv)^2] \cdot \\ & \cdot \frac{(1-(uv)^{2(g-1)+d-k+1}) \cdot \ldots \cdot (1-(uv)^{2(g-1)+d})}{(1-uv) \cdot \ldots \cdot (1-(uv)^k)}.\end{aligned}$$

In order to compute $\mathcal{H}(\mathscr{W}_{\mathcal{E}_{\underline{n}}})(u,v)$, $\mathcal{H}(\mathscr{W}_{\mathcal{E}'_{\underline{n}}})(u,v)$, $\mathcal{H}(\mathscr{W}_{\mathcal{ES}_{\underline{n}}})(u,v)$ and $\mathcal{H}(\mathscr{W}_{\mathcal{ES}'_{\underline{n}}})(u,v)$ we use Theorem 8.17. These cases are easier. We do not need to take into account the action of a group, because we can do the construction as complements of determinantal varieties at the moduli space level.

Note that we can describe our strata as locally trivial fiber bundles (Remark 7.9). The fiber is the complement in a Grassmannian of a union of certain varieties as one can see in the proof of Theorem 8.17. The base space in the different locally trivial fiber bundles is the space classifying the different types of extensions that can appear in the case we are dealing with, see Proposition 6.14. We use here the notation of Section 7.2.

Let $\mathcal{E}_{\underline{n}}$ be the space that parametrizes the extensions

$$0 \to L \to F \to L' \to 0$$

that are nonsplit and such that the line bundles L and L' are nonisomorphic. Let $\mathcal{E}'_{\underline{n}}$ be the space that parametrizes the extensions as above where $L \cong L'$.

Now, $\mathscr{W}_{\mathcal{E}_{\underline{n}}}$ and $\mathscr{W}_{\mathcal{E}'_{\underline{n}}}$ we can describe as locally trivial fiber bundles over $\mathcal{E}_{\underline{n}}$ and $\mathcal{E}'_{\underline{n}}$ respectively (Remark 7.9). The fiber is the same in both cases (see Theorems 7.5 and 7.7) and this is given in Theorem 8.17.

Then

$$\mathcal{H}(\mathscr{W}_{\mathcal{E}_{\underline{n}}})(u,v)+\mathcal{H}(\mathscr{W}_{\mathcal{E}'_{\underline{n}}})(u,v)=\big(\mathcal{H}(\mathcal{E}_{\underline{n}})(u,v)+\mathcal{H}(\mathcal{E}'_{\underline{n}})(u,v)\big)\cdot$$
$$\cdot\Bigg[\frac{(1-(uv)^{2(g-1)+d-k+1})\cdot\ldots\cdot(1-(uv)^{2(g-1)+d})}{(1-uv)\cdot\ldots\cdot(1-(uv)^k)}-$$
$$-\sum_{\mu=\lceil\frac{k}{2}\rceil}^{\min\{k,(g-1)+\frac{d}{2}\}}(u\cdot v)^{\mu(d/2+(g-1)-k+\mu)}\cdot\frac{(1-(uv)^{d/2+(g-1)-k+\mu+1})\cdot\ldots\cdot(1-(uv)^{d/2+g-1})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{k-\mu})}\cdot$$
$$\cdot\frac{(1-(uv)^{(g-1)+d/2-\mu+1})\cdot\ldots\cdot(1-(uv)^{(g-1)+d/2})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{\mu})}\Bigg]$$

We saw in Proposition 6.14 (ii) that $\mathcal{E}_{\underline{n}}$ is a projective bundle over $\mathrm{Jac}^{d/2}X\times\mathrm{Jac}^{d/2}X\setminus\Delta$, where Δ is the diagonal in $\mathrm{Jac}^{d/2}X\times\mathrm{Jac}^{d/2}X$, with fiber the projective space of dimension $g-2$. The Hodge–Deligne polynomials of the Jacobian and the projective space are:

$$\mathcal{H}(\mathrm{Jac}^{\delta}X)(u,v)=(1+u)^g(1+v)^g\quad\text{and}\quad\mathcal{H}(\mathbb{P}^n)(u,v)=\frac{1-(uv)^{n+1}}{1-uv},\tag{68}$$

for every degree δ. Then, using Lemma 8.12 we get

$$\mathcal{H}(\mathcal{E}_{\underline{n}})(u,v)=((1+u)^{2g}(1+v)^{2g}-(1+u)^g(1+v)^g)\cdot\frac{1-(uv)^{g-1}}{1-uv}.$$

Again, by Proposition 6.14 (ii) $\mathcal{E}'_{\underline{n}}$ is a projective bundle over $\mathrm{Jac}^{d/2}X$ with fiber the projective space of dimension $g-1$, so its Hodge–Deligne polynomial is

$$\mathcal{H}(\mathcal{E}'_{\underline{n}})(u,v)=(1+u)^g(1+v)^g\cdot\frac{1-(uv)^g}{1-uv}.$$

For the spliting cases, $\mathcal{SE}_t$ parametrizes the split extensions

$$0\to L\to L\oplus L'\to L'\to 0$$

where L and L' are nonisomorphic line bundles of the same degree $d/2$. By Paragraph 6.17 these are classify by $(\mathrm{Jac}^{d/2}X\times\mathrm{Jac}^{d/2}X\setminus\Delta)/(\mathbb{Z}/2)$ where $\mathbb{Z}/2$ is acting on $\mathrm{Jac}^{d/2}X\times\mathrm{Jac}^{d/2}X\setminus\Delta$ by permuting the two factors. When M is a smooth projective variety, Muñoz *et al.* (see [MOV2] Lemma 2.6), compute the Hodge–Deligne polynomial of $(M\times M)/(\mathbb{Z}/2)$, this is

$$\mathcal{H}\big((M\times M)/(\mathbb{Z}/2)\big)(u,v)=\frac{1}{2}\big(\mathcal{H}(M)(u,v)^2+\mathcal{H}(M)(-u^2,-v^2)\big).$$

Combining the latter and Theorem 8.11 we obtain

$$\mathcal{H}(\mathcal{SE}_{\underline{n}})(u,v)=\mathcal{H}\Big((\mathrm{Jac}^{d/2}X\times\mathrm{Jac}^{d/2}X\setminus\Delta)/(\mathbb{Z}/2)\Big)(u,v)=$$
$$=\frac{1}{2}\Big[(1+u)^{2g}(1+v)^{2g}+(1-u^2)^g(1-v^2)^g\Big]-(1+u)^g(1+v)^g.$$

The stratum $\mathscr{W}_{\mathcal{SE}_{\underline{n}}}$ can be identified with a locally trivial fibration over $\mathcal{SE}_{\underline{n}}$ where the fiber can be described as follows. In Theorem 7.6 we saw that for every point $w=(L,L')\in\mathrm{Jac}^{d/2}X\times\mathrm{Jac}^{d/2}X\setminus\Delta$ we define the determinantal varieties

$$V^1_w:=\big\{\pi\in\mathrm{Gr}(k,2(g-1)+d):\dim\big(\pi\cap H^1(L'^{\vee})\big)\geqslant\frac{k}{2}\big\},$$

and

$$V^2_w:=\big\{\pi\in\mathrm{Gr}(k,2(g-1)+d):\dim\big(\pi\cap H^1(L^{\vee})\big)\geqslant\frac{k}{2}\big\}.$$

Then, for every point $w = (L, L') \in \mathcal{SE}_{\underline{n}}$ the fiber of the bundle $\mathscr{W}_{\mathcal{SE}_{\underline{n}}}$ is the complement in $\mathrm{Gr}(k, 2(g-1)+d)$ of $V_w^1 \cup V_w^2$. Note that this does not depend on w.

From (62) we have that V_w^1 can be written as

$$V_w^1 := \coprod_{\mu=\lceil k/2\rceil}^{\min\{k,g-1+d/2\}} \left\{\pi \in \mathrm{Gr}(k, 2(g-1)+d) : \dim\left(\pi \cap H^1(L'^\vee)\right) = \mu\right\},$$

and the same is true for V_w^2. We denote $V_1^\mu := \left\{\pi \in \mathrm{Gr}(k, 2(g-1)+d) : \dim\left(\pi \cap H^1(L'^\vee)\right) = \mu\right\}$ for integers μ between $\lceil k/2\rceil$ and $\min\{k, g-1+d/2\}$. Analogously for V_2^μ.

By (63) the varieties V_i^μ are isomorphic to a fibration over $\mathrm{Gr}(k-\mu, g-1+d/2) \times \mathrm{Gr}(\mu, g-1+d/2)$ with fibre $\mathbb{C}^{(g-1+d/2-\mu)(k-\mu)}$. Then

$$\mathcal{H}(V_i^\mu)(u,v) = \mathcal{H}(\mathrm{Gr}(k-\mu, g-1+d/2))(u,v) \cdot \mathcal{H}(\mathrm{Gr}(\mu, g-1+d/2))(u,v) \cdot \mathcal{H}(\mathbb{C}^{(g-1+d/2-\mu)(k-\mu)})(u,v).$$

Note that the varieties V_i^μ may have non-empty intersection. That would be only possible when k is even, in that case $V_w^1 \cap V_w^2 \subset V_i^{k/2}$, and $V_w^1 \cap V_w^2$ is parametrized by

$$\mathrm{Gr}(k/2, g-1+d/2) \times \mathrm{Gr}(k/2, g-1+d/2).$$

Then, the Hodge–Deligne polynomial of $\mathscr{W}_{\mathcal{SE}_{\underline{n}}}$ is given by

$$\begin{aligned}
&\mathcal{H}(\mathscr{W}_{\mathcal{SE}_{\underline{n}}})(u,v) = \mathcal{H}(\mathcal{SE}_{\underline{n}})(u,v) \cdot \big[\mathcal{H}(\mathrm{Gr}(k, 2(g-1)+d))(u,v) - \mathcal{H}(V_w^1)(u,v) - \mathcal{H}(V_w^2)(u,v) + \\
&\quad + \mathcal{H}(V_w^1 \cap V_w^2)(u,v)\big] = \mathcal{H}(\mathcal{SE}_{\underline{n}})(u,v) \cdot \left[\frac{(1-(uv)^{2(g-1)+d-k+1}) \cdot \ldots \cdot (1-(uv)^{2(g-1)+d})}{(1-uv) \cdot \ldots \cdot (1-(uv)^k)} - \right. \\
&\quad - 2 \sum_{\mu=\frac{k}{2}}^{\min\{k,(g-1)+\frac{d}{2}\}} (u \cdot v)^{\mu(d/2+(g-1)-k+\mu)} \cdot \frac{(1-(uv)^{d/2+(g-1)-k+\mu+1}) \cdot \ldots \cdot (1-(uv)^{d/2+g-1})}{(1-uv) \cdot \ldots \cdot (1-(uv)^{k-\mu})} \cdot \\
&\quad \cdot \frac{(1-(uv)^{(g-1)+d/2-\mu+1}) \cdot \ldots \cdot (1-(uv)^{(g-1)+d/2})}{(1-uv) \cdot \ldots \cdot (1-(uv)^\mu)} + \\
&\quad \left. + \frac{(1-(uv)^{d/2+(g-1)-k/2+1})^2 \cdot \ldots \cdot (1-(uv)^{d/2+g-1})^2}{(1-uv)^2 \cdot \ldots \cdot (1-(uv)^{k/2})^2}\right]
\end{aligned}$$

when k is even, and

$$\begin{aligned}
&\mathcal{H}(\mathscr{W}_{\mathcal{SE}_{\underline{n}}})(u,v) = \mathcal{H}(\mathcal{SE}_{\underline{n}})(u,v) \cdot \left[\frac{(1-(uv)^{2(g-1)+d-k+1}) \cdot \ldots \cdot (1-(uv)^{2(g-1)+d})}{(1-uv) \cdot \ldots \cdot (1-(uv)^k)} - \right. \\
&\quad - 2 \sum_{\mu=\lceil\frac{k}{2}\rceil}^{\min\{k,(g-1)+\frac{d}{2}\}} (u \cdot v)^{\mu(d/2+(g-1)-k+\mu)} \cdot \frac{(1-(uv)^{d/2+(g-1)-k+\mu+1}) \cdot \ldots \cdot (1-(uv)^{d/2+g-1})}{(1-uv) \cdot \ldots \cdot (1-(uv)^{k-\mu})} \cdot \\
&\quad \left. \cdot \frac{(1-(uv)^{(g-1)+d/2-\mu+1}) \cdot \ldots \cdot (1-(uv)^{(g-1)+d/2})}{(1-uv) \cdot \ldots \cdot (1-(uv)^\mu)}\right]
\end{aligned}$$

when k is odd.

Finally, the space $\mathcal{SE}'_{\underline{n}}$ parametrizes the split extensions in which the bundle in the middle is the direct sum of two copies of the same line bundle of degree $d/2$. These are classified by $\mathrm{Jac}^{d/2} X$.

The stratum $\mathscr{W}_{\mathcal{SE}'_{\underline{n}}}$ can be identified with a locally trivial fibration over $\mathcal{SE}'_{\underline{n}}$ where the fiber can be described as follows. For every point $w = (L, f) \in \mathrm{Jac}^{d/2} X \times \mathbb{P}^1$ we define the determinantal variety

$$V_L^f := \left\{\pi \in \mathrm{Gr}(k, 2(g-1)+d) : \dim\left(\pi \cap \ker H^1(f^\vee)\right) \geqslant \frac{k}{2}\right\}.$$

For every $L \in \mathrm{Jac}^{d/2}X$ we define a new variety W_L as a locally trivial fibration over $\mathbb{P}^1$, $\pi : W_L \to \mathbb{P}^1$ whose fiber at a point $f \in \mathbb{P}^1$ is $\pi^{-1}(\{f\}) = V_L^f$. Then, $\mathscr{W}_{\mathcal{SE}'_{\underline{n}}}$ can be identified with a locally trivial fibration over $\mathcal{SE}'_{\underline{n}}$ whose fiber at a point $L \in \mathcal{SE}'_{\underline{n}} \cong \mathrm{Jac}^{d/2}X$ is $\mathrm{Gr}(k, 2(g-1)+d) \setminus W_L$. Note that W_L does not depend on L. We denote $W = W_L$. Then, by Theorem 8.11 and Lemma 8.12 one has that

$$\begin{aligned}\mathcal{H}(\mathscr{W}_{\mathcal{SE}'_{\underline{n}}})(u,v) &= \mathcal{H}(\mathcal{SE}'_{\underline{n}})(u,v) \cdot (\mathcal{H}(\mathrm{Gr}(k,2(g-1)+d))(u,v) - \mathcal{H}(W)(u,v)) = \\ &= \mathcal{H}(\mathrm{Jac}^{d/2}X)(u,v) \cdot (\mathcal{H}(\mathrm{Gr}(k,2(g-1)+d))(u,v) - \mathcal{H}(W)(u,v)) = \\ &= (1+u)^g(1+v)^g \left[\frac{(1-(uv)^{2(g-1)+d-k+1}) \cdot \ldots \cdot (1-(uv)^{2(g-1)+d})}{(1-uv) \cdot \ldots \cdot (1-(uv)^k)} - \mathcal{H}(W)(u,v)\right].\end{aligned} \tag{69}$$

Regarding $\mathcal{H}(W)(u,v)$, from Lemma 8.12 and (68) we have that

$$\mathcal{H}(W)(u,v) = \mathcal{H}(\mathbb{P}^1)(u,v) \cdot \mathcal{H}(V_L^f)(u,v) = (1+uv) \cdot \mathcal{H}(V_L^f)(u,v). \tag{70}$$

Now, from (62) one gets

$$V_L^f := \coprod_{\mu=\lceil k/2 \rceil}^{\min\{k, g-1+d/2\}} (V_L^f)^\mu \tag{71}$$

where $(V_L^f)^\mu = \{\pi \in \mathrm{Gr}(k, 2(g-1)+d) : \dim\left(\pi \cap \ker H^1(f^\vee)\right) = \mu\}$. By (63) the variety $(V_L^f)^\mu$, for every μ, is isomorphic to a fibration over $\mathrm{Gr}(k-\mu, g-1+d/2) \times \mathrm{Gr}(\mu, g-1+d/2)$ with fibre $\mathbb{C}^{(g-1+d/2-\mu)(k-\mu)}$. Then

$$\mathcal{H}((V_L^f)^\mu)(u,v) = \mathcal{H}(\mathrm{Gr}(k-\mu, g-1+d/2))(u,v) \cdot \mathcal{H}(\mathrm{Gr}(\mu, g-1+d/2))(u,v) \cdot \mathcal{H}(\mathbb{C}^{(g-1+d/2-\mu)(k-\mu)})(u,v).$$

If we fix $L \in \mathrm{Jac}^{d/2}X$ and we allow $f \in \mathbb{P}^1$ to vary, one has that the varieties $\{(V_L^f)^\mu\}_f$ may have non-empty intersection. That would be only possible when k is even. Note that for k even it could be possible to find a $\pi \in \mathrm{Gr}(k, g-1+d/2)$ such that $\dim \pi \cap \ker H^1(f^\vee) = k/2$ and $\dim \pi \cap \ker H^1(g^\vee) = k/2$ and $\pi = (\pi \cap \ker H^1(f^\vee)) \oplus (\pi \cap \ker H^1(g^\vee))$ for some $f, g \in \mathbb{P}^1$. Note also that if such an f exists, then there is at most one g that satisfies the conditions above. Let $I := \{\pi \in \mathrm{Gr}(k, g-1+d/2)$ such that $\dim \pi \cap \ker H^1(f^\vee) = k/2$ and $\dim \pi \cap \ker H^1(g^\vee) = k/2$ and $\pi = (\pi \cap \ker H^1(f^\vee)) \oplus (\pi \cap \ker H^1(g^\vee))$ for some $f, g \in \mathbb{P}^1\}$. Then, I is parametrized by

$$(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathbb{P}^1) \times \mathrm{Gr}(k/2, g-1+d/2) \times \mathrm{Gr}(k/2, g-1+d/2).$$

Hence, the Hodge–Deligne polynomial of W is

$$\begin{aligned}\mathcal{H}(W)(u,v) &= \mathcal{H}(\mathbb{P}^1)(u,v) \cdot \mathcal{H}(V_L^f)(u,v) - \mathcal{H}(I)(u,v) = (1+uv) \cdot \mathcal{H}(V_L^f)(u,v) - \mathcal{H}(I)(u,v) = \\ &= (1+uv) \cdot \left[\sum_{\mu=k/2}^{\min\{k,g-1+d/2\}} \mathcal{H}((V_L^f)^\mu)(u,v)\right] - \mathcal{H}(I)(u,v) = \\ &= (1+uv) \cdot \left[\sum_{\mu=k/2}^{\min\{k,g-1+d/2\}} \mathcal{H}(\mathrm{Gr}(k-\mu, g-1+d/2))(u,v) \cdot \mathcal{H}(\mathrm{Gr}(\mu, g-1+d/2))(u,v) \cdot \right. \\ &\left. \cdot \mathcal{H}(\mathbb{C}^{(g-1+d/2-\mu)(k-\mu)})(u,v)\right] - (\left[\mathcal{H}(\mathbb{P}^1)(u,v)\right]^2 - \mathcal{H}(\mathbb{P}^1)(u,v))\left[\cdot \mathcal{H}(\mathrm{Gr}(k/2, g-1+d/2))(u,v)\right]^2\end{aligned} \tag{72}$$

when k is even, and

$$\mathcal{H}(W)(u,v) = \mathcal{H}(\mathbb{P}^1)(u,v) \cdot \mathcal{H}(V_L^f)(u,v) = (1+uv) \cdot \mathcal{H}(V_L^f)(u,v) = \tag{73}$$

$$= (1+uv)\cdot\Bigg[\sum_{\mu=\lceil k/2\rceil}^{\min\{k,g-1+d/2\}} \mathcal{H}((V_L^f)^{\mu})(u,v)\Bigg] =$$

$$= (1+uv)\cdot\Bigg[\sum_{\mu=\lceil k/2\rceil}^{\min\{k,g-1+d/2\}} \mathcal{H}(\mathrm{Gr}(k-\mu,g-1+d/2))(u,v)\cdot\mathcal{H}(\mathrm{Gr}(\mu,g-1+d/2))(u,v)\cdot$$

$$\cdot\mathcal{H}(\mathbb{C}^{(g-1+d/2-\mu)(k-\mu)})(u,v)\Bigg]$$

when k is odd.

Then, from (69) and (72) the Hodge–Deligne polynomial of $\mathscr{W}_{\mathcal{SE}'_{\underline{n}}}$ is given by

$$\mathcal{H}(\mathscr{W}_{\mathcal{SE}'_{\underline{n}}})(u,v) = (1+u)^g(1+v)^g\cdot\Bigg[\frac{(1-(uv)^{2(g-1)+d-k+1})\cdot\ldots\cdot(1-(uv)^{2(g-1)+d})}{(1-uv)\cdot\ldots\cdot(1-(uv)^k)}-$$

$$-(1+uv)\sum_{\mu=\frac{k}{2}}^{\min\{k,(g-1)+\frac{d}{2}\}}(u\cdot v)^{\mu(d/2+(g-1)-k+\mu)}\cdot\frac{(1-(uv)^{d/2+(g-1)-k+\mu+1})\cdot\ldots\cdot(1-(uv)^{d/2+g-1})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{k-\mu})}\cdot$$

$$\cdot\frac{(1-(uv)^{(g-1)+d/2-\mu+1})\cdot\ldots\cdot(1-(uv)^{(g-1)+d/2})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{\mu})}+$$

$$+((1+uv)^2-(1+uv))\cdot\frac{(1-(uv)^{d/2+(g-1)+k/2+1})^2\cdot\ldots\cdot(1-(uv)^{d/2+g-1})^2}{(1-uv)^2\cdot\ldots\cdot(1-(uv)^{k/2})^2}\Bigg]$$

when k is even, and from (69) and (73) one obtains

$$\mathcal{H}(\mathscr{W}_{\mathcal{SE}'_{\underline{n}}})(u,v) = (1+u)^g(1+v)^g\cdot\Bigg[\frac{(1-(uv)^{2(g-1)+d-k+1})\cdot\ldots\cdot(1-(uv)^{2(g-1)+d})}{(1-uv)\cdot\ldots\cdot(1-(uv)^k)}-$$

$$-(1+uv)\sum_{\mu=\lceil\frac{k}{2}\rceil}^{\min\{k,(g-1)+\frac{d}{2}\}}(u\cdot v)^{\mu(d/2+(g-1)-k+\mu)}\cdot\frac{(1-(uv)^{d/2+(g-1)-k+\mu+1})\cdot\ldots\cdot(1-(uv)^{d/2+g-1})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{k-\mu})}\cdot$$

$$\cdot\frac{(1-(uv)^{(g-1)+d/2-\mu+1})\cdot\ldots\cdot(1-(uv)^{(g-1)+d/2})}{(1-uv)\cdot\ldots\cdot(1-(uv)^{\mu})}\Bigg]$$

when k is odd.

Summing up all the previous polynomials together we obtain the result. □

Remark 8.21. Note that for given (n,d,k) satisfying that $(n-k,d)=(2,d)\neq 1$ and k being odd, one immediately obtains that $(n,d,k)=1$. Under this condition, the moduli space of α_L-stable coherent systems, $G_L(n,d,k)$, is projective, smooth and irreducible (see [KN] and Proposition 4.5). Then, from remark 8.10 one can obtain the usual Poincaré polynomial of $G_L(n,d,k)$ just by knowing its Hodge–Deligne polynomial. Hence, the second formula of Theorem 8.20 allows us to compute the Poincaré polynomial of $G_L(n,d,k)$ when $(n,d,k)=1$ and $(n-k,d)=(2,d)\neq 1$.

Acknowledgements. Acknowledgements to be written.

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